



Applying the combined integral method to one-dimensional ablation

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ABSTRACT

In this paper the combined integral method is applied to a simple one-dimensional ablation problem. One of the drawbacks of heat balance integral methods is how to choose the approximating function. It is common to use a polynomial form but even then it is not clear what the power of the highest order term should be. Previous studies have determined exponents either from exact solutions or from expansions valid over short time scales; neither approach is satisfactory nor very accurate for larger times. We combine the heat balance and refined integral methods to determine this exponent as part of the solution process, and conclude that it is in fact time-dependent in the ablation stage. From comparing the approximate solutions with numerical and exact analytical solutions whenever possible, we show that this new method greatly improves the accuracy on standard methods, without overcomplicating the method.

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1. Introduction

Ablation is defined as the removal of a material from the surface of an object by vaporisation, chipping or other erosive processes. Classic examples include heat shields on space vehicles or the burning up of meteorites [1,2]. On space vehicles the exposed surface of the ablative material is designed to burn off and the resultant gases will carry much of the heat away, while the remaining material acts as an insulator. The process may therefore be considered as a one-phase Stefan problem. Other applications of ablation are in the melting or sublimation of a solid [3], laser drilling in metals and the cornea [4], the reduction of glaciers by erosion [5] and the surgical removal of a body part or tissue, such as in atrial fibrillation [6].

In this paper we are concerned with applying a combination of conventional heat balance and refined integral methods to the ablation problem. Our aim is to develop an integral method that provides a more accurate description of the process of ablation. The main advantage of these integral methods is that they significantly reduce the complexity of the problem. The process is governed by a heat equation coupled to a Stefan condition that gives the domain over which the heat equation is applied, in this case $x \in [s, \delta]$. The standard heat balance integral method begins by introducing a heat penetration depth, $\delta(t)$. For $x \geq \delta$ the temperature change above the initial temperature is assumed to be negligible. Then an approximate function is defined for the temperature, typically a polynomial, and by applying sufficient boundary conditions at $x = s(t)$ and $\delta(t)$, all the unknown coefficients can be determined in terms of the unknown function δ and the unknown ablation depth $s(t)$. Finally, the governing heat equation is integrated for $x \in [s, \delta]$ to produce a heat balance integral, and this is coupled to the Stefan condition leading to a pair of ordinary differential equations for s and δ . The problem is therefore reduced to solving two first-order differential equations in time. An alternative approach to the HBIM, developed by Sadoun et al. [7], is the refined integral method (RIM), where the heat equation is integrated twice. The relative merits of the two approaches, as well as a number of variations involving alternative approximating functions, are described in detail in Mitchell and Myers [8].

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Nomenclature

$a(t)$	coefficient in logarithmic profile (18)
$b(t)$	coefficient in logarithmic profile (18)
E_n	Langford error (10)
$f(t)$	time-dependent boundary condition
$h(t)$	position of turning point
k	thermal conductivity
L	lengthscale
L_m	latent heat of ablation
n	exponent of approximating function in preablation stage
p	exponent of approximating function in ablation stage
Q	heat flux
s	nondimensional ablation depth
s'	ablation depth
t	nondimensional time
t'	time
t_1	time at end of pre-ablation stage
T	nondimensional temperature
T'	temperature
T_0	initial temperature
T_a	ablation temperature
x	nondimensional distance from ablation boundary
x'	distance from ablation boundary
α	coefficient of $\delta(= \sqrt{\alpha t})$ in pre-ablation stage
β	$\kappa \rho L_m / (k \Delta T)$, inverse Stefan number
δ	nondimensional heat penetration depth
δ'	heat penetration depth
κ	thermal diffusivity
ρ	density
τ	timescale

Goodman [9] applied the standard HBIM with a polynomial approximating function to this problem and compared results with a numerical solution of Landau [10], whilst Zien [11] applied an exponential approximation. More recently Blackwell and Hogan [12] have solved the problem numerically, using an explicit finite difference scheme, and Mitchell and Vynnycky [13] have developed a fully implicit second order accurate finite difference scheme. Braga et al. [14,15] adapted the standard HBIM approach so that the approximating temperature function is a polynomial of the form $T = a_0 + a_n(\delta - x)^n$. The non-integer order n for the pre-ablation stage is determined from the exact analytical solution. The choice of the power of the highest order term in the approximating polynomial is the most contentious aspect of the HBIM. Recently, Mitchell and Myers [16,17] have developed a method where the exponent is determined during the solution process, producing significantly better results than all previous models. It involves a combination of the conventional heat balance and refined integral methods, and is called the *combined integral method* (CIM). The method has been applied to both standard thermal problems [16] and standard Stefan problems [17]. In addition, Myers [18,19] has employed an alternative method to determine the exponent by minimising an error function. This also gives a great improvement on the accuracy but the minimisation does not easily allow for situations when the exponent is time-dependent.

Mitchell and Myers [16] also found that an extra improvement can be made by including a logarithmic term in the approximating function. This has most benefit for thermal problems with time-dependent boundary conditions, since heat balance methods with polynomial profiles are only possible for small time or very limited (and not physically realistic) forms of the time-dependent function, see [14,13,7,11] for example. The logarithmic profile proves to be very effective in dealing with more complicated boundary conditions, such as those decreasing monotonically with time. Goodman [9] also noted that the heat balance integral method is only useful for time-dependent boundary conditions that are monotonically increasing or constant.

Mitchell and Myers [20] applied the standard HBIM to the ablation problem with $n = 4$ taken to be the polynomial exponent. This was motivated by an expansion of an error function solution to the pre-ablation stage, i.e., where the material is heated up to the ablation temperature. The same value of n was then used during the ablation stage. Although this gave more accurate results than previous studies, it did not capture the behaviour well for larger times, since the $n = 4$ assumption was only really valid in the small-time pre-ablation stage. We demonstrate here that once ablation starts, the exponent is in fact *time-dependent*, and allowing this in the model leads to solutions that are much closer to a very accurate numerical solution [13]. Mitchell and Myers [8] gave a review of the ablation problem and compared the HBIM, RIM and ARIM formulations, the latter being an alternative integral formulation to the RIM. Different polynomial profiles were considered for each of the

approximate methods and although it could not be concluded that one method is best in both the pre-ablation and ablation stages, the RIM was always the worst method. Two main advantages of the CIM are that, firstly, it avoids the need to choose between the different integral formulations and, secondly, it is generally far more accurate than any of the HBIM, RIM or ARIM.

The layout of the paper is as follows. In the next section we describe the problem statement and governing equations. In Section 3 the CIM is applied to the pre-ablation stage, and results are compared with an analytical solution as well as the standard HBIM. We also discuss the improvement that can be made using the logarithmic profile, as described above. Section 4 considers the ablation stage; by comparison with a numerical solution we show that the CIM, which has a time-dependent exponent in the approximating function, gives far more accurate results than the standard HBIM method with a constant exponent. Finally in Section 5 we draw conclusions.

2. The mathematical model

The typical configuration is shown in Fig. 1. The problem is governed by the heat equation:

$$\kappa \frac{\partial^2 T'}{\partial x'^2} = \frac{\partial T'}{\partial t'}, \tag{1}$$

and Stefan condition:

$$\rho L_m \frac{ds'}{dt'} = Q + k \frac{\partial T'}{\partial x'}, \quad \text{at } x' = s', \tag{2}$$

where κ is the thermal diffusivity, ρ is the density, k is the thermal conductivity and L_m is the latent heat of melting. Initially the ablator is at a temperature $T(x',0) = T_0$ and the surface $x' = 0$ is subject to a constant heat flux Q , namely $T'_{x'}(0, t') = -Q/k$. There follows a heating up stage where heat penetrates the material and the surface $x' = 0$ reaches the ablation temperature T_a ; subsequently material is removed at the ablating interface, $x' = s'(t')$, and we need only solve for the temperature in the solid which occupies $s'(t') \leq x' \leq H$. Ablation differs to similar Stefan problems, such as the two-phase melting of a semi-infinite solid, in that the process only involves a solid phase. For simplicity we restrict attention to the semi-infinite problem $x' \rightarrow H$, therefore allowing the process to occur in two distinct stages, as discussed below. The finite problem has been considered by Mitchell and Myers [8,20] where it was assumed that the surface $x' = H$ is insulated, giving $T'_{x'}(H, t') = 0$.

The system is non-dimensionalised by setting:

$$x = \frac{x'}{L}, \quad t = \frac{t'}{\tau}, \quad s = \frac{s'}{L}, \quad T = \frac{T' - T_a}{\Delta T}, \tag{3}$$

where $\Delta T = T_a - T_0$. The timescale is from the heat Eq. (1), giving $\tau = L^2/\kappa$, and the lengthscale comes from balancing the right hand side of the Stefan condition (2), giving $L = k\Delta T/Q$.

The two stages of the process are described as follows. For $t \in [0, t_1]$ the ablator is heated, and this stage ends when $T(0, t) = 0$. In the second stage ablation occurs and the heat penetrates through the solid material. During this stage there is always a region (sufficiently far from $x = 0$) where the temperature is at the initial temperature, $T(x, t) = -1$. For the finite problem there would be a time t_2 such that for $t \geq t_2$ the remaining material is all above the initial temperature and continues to heat up until it has all ablated.

The equations governing the two stages for the semi-infinite problem are:

- **Stage 1**, with $t \in [0, t_1]$ and $s(t) = 0$:

$$\frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t}, \quad \frac{\partial T}{\partial x}(0, t) = -1, \quad T \rightarrow -1 \quad \text{as } x \rightarrow \infty, \tag{4}$$

for $x > 0$. The stage ends when $T(0, t_1) = 0$ and this defines t_1 . The initial condition is $T(x, 0) = -1$.

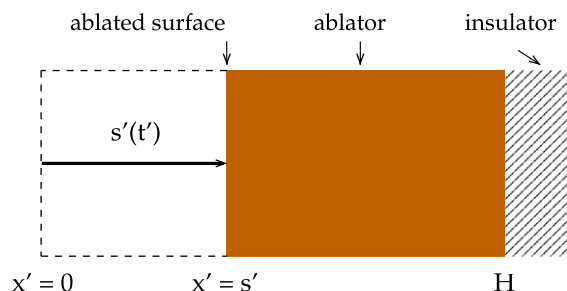


Fig. 1. Typical ablator configuration.

Table 1
Physical parameter values for Teflon.

Parameter	Typical value	Units
c	1256	J/kg K
k	0.22	W/m K
ρ	1922	kg/m ³
κ	9.11×10^{-8}	m ² /s
L_m	2.326×10^6	J/kg
T_a	833	K
T_0	273	K

• **Stage 2**, with $t > t_1$:

$$\frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t}, \quad T(s, t) = 0, \quad T \rightarrow -1 \quad \text{as } x \rightarrow \infty, \quad (5)$$

for $x > s(t)$. The external heat input now appears in the Stefan condition:

$$\beta \frac{ds}{dt} = 1 + \frac{\partial T}{\partial x}, \quad \text{at } x = s, \quad (6)$$

where $\beta = \kappa \rho L_m / (k \Delta T)$ is the inverse Stefan number. The initial conditions are $s(t_1) = 0$ and $T(x, t_1)$ is defined from stage 1.

Typical parameter values for Teflon are given in Table 1, which are taken from [20]. Using these values gives $\beta \sim 3.31$ but in the results shown below we use $\beta = 1$, $\beta = 5$ and $\beta = 0.1$ to allow for different types of material (for example $\beta \sim 0.37$ for copper and $\beta \sim 0.002$ for silicon-dioxide). It is well known that for classic Stefan problems the accuracy of heat balance integral methods varies widely for different values of β [8,17]. We can therefore also check if this is true for the ablation problem.

We now apply the CIM to both stages of the ablation process. In the pre-ablation stage there is an exact solution which we can use to compare with our approximate solution. In the ablation stage we check the accuracy by comparison with an implicit second order accurate numerical method developed by Mitchell and Vynnycky [13].

3. Pre-ablation stage

Consider the problem given in (4). The exact solution is:

$$T(x, t) = -1 + 2\sqrt{\frac{t}{\pi}} \exp\left(-\frac{x^2}{4t}\right) - x \operatorname{erfc}\left(\frac{x}{2\sqrt{t}}\right). \quad (7)$$

As mentioned in the introduction, for the above initial boundary value problem the HBIM involves three steps:

- (i) First we define the heat penetration depth, $\delta(t)$. For $x \geq \delta$ the temperature change from the initial temperature is negligible and hence $T(\delta, t) = -1$, $T_x(\delta, t) = 0$.
- (ii) An approximating function is then introduced, typically a polynomial, which is valid over the domain $0 \leq x \leq \delta(t)$. The unknown coefficients are determined from the boundary conditions.
- (iii) Finally, the heat balance integral is derived from integrating the heat equation over $x \in [0, \delta]$. This results in a single ordinary differential equation for δ , which can often be solved analytically.

As mentioned in the Introduction Sadoun et al. [7] developed an alternative method, known as the refined integral method (RIM), which simply involves carrying out a double integration of the heat equation at stage (iii).

The standard approximating polynomial for the HBIM has the form:

$$T = -1 + \frac{\delta}{n} \left(1 - \frac{x}{\delta}\right)^n, \quad (8)$$

which satisfies $T_x(0, t) = -1$ and the far-right condition $T_x(\delta, t) = 0$ provided $n > 1$. An extra condition is often applied, which is derived by noting:

$$\frac{DT}{Dt}(\delta, t) = \left(\frac{\partial T}{\partial t} + \frac{\partial T}{\partial x} \frac{d\delta}{dt}\right)_{x=\delta} = 0 \Rightarrow \frac{\partial^2 T}{\partial x^2}(\delta, t) = 0, \quad (9)$$

using the heat equation $T_t = T_{xx}$ and boundary condition $T_x(\delta, t) = 0$. Imposing condition (9) requires $n > 2$. In general $n = 2$ is usually chosen as it provides the simplest profile that satisfies the boundary conditions.

To improve the accuracy of the HBIM solution Myers [18,19] left n to be determined as part of the solution process. The profile (8) therefore involves two unknowns, the heat penetration depth $\delta(t)$ and the exponent n , which was chosen to minimise the error defined by Langford [21]:

$$E_n = \int_0^\delta \left[\frac{\partial T}{\partial t} - \frac{\partial^2 T}{\partial x^2} \right]^2 dx \geq 0. \tag{10}$$

For this particular problem it was shown in [18] that the error is minimised by choosing $n \approx 3.584$.

Although the minimisation technique gives significantly more accurate results than the standard HBIM approach, the integral in (10) can be quite complex. However for standard problems one can simply use the values of n quoted in [18,19]. The main disadvantage of this approach is that for certain boundary conditions the exponent n is time-dependent, the integral in E_n then involves time derivatives in n which cannot be evaluated. Mitchell and Myers [16] developed the combined integral method (CIM) to obtain n , and this is less algebraically complex than the minimisation technique and allows n to vary with t if necessary. The method is a combination of the HBIM and RIM and we now apply it to the standard thermal problem.

The standard heat balance integral is determined by integrating the heat equation over the interval $[0, \delta]$. This gives:

$$\frac{d}{dt} \int_0^\delta T dx + \frac{d\delta}{dt} = -\frac{\partial T}{\partial x}(0, t), \tag{11}$$

where we have taken the time derivative outside of the integral and applied the boundary conditions $T_x(\delta, t) = 0$ and $T(\delta, t) = -1$. For the RIM we integrate the heat equation twice with respect to x :

$$\int_0^\delta \left(\int_0^x \frac{\partial T}{\partial t} d\xi \right) dx = \int_0^\delta \left(\frac{\partial T}{\partial x} - \frac{\partial T}{\partial x}(0, t) \right) dx,$$

and then integrate the left hand side by parts to obtain:

$$\delta \int_0^\delta \frac{\partial T}{\partial t} dx - \int_0^\delta x \frac{\partial T}{\partial t} dx = -T(0, t) - \delta \frac{\partial T}{\partial x}(0, t), \tag{12}$$

where for convenience we have relabelled ξ as x in the first integral. Since $T_t = T_{xx}$ this term can be integrated again yielding the RIM equation:

$$\frac{d}{dt} \int_0^\delta xT dx + \delta \frac{d\delta}{dt} = T(0, t). \tag{13}$$

Substituting T from Eq. (8) into (11) and (13) leads to the following expressions:

$$\text{HBIM : } \frac{d}{dt} \left[\frac{\delta^2}{n(n+1)} \right] = 1, \quad \text{RIM : } \frac{d}{dt} \left[\frac{\delta^3}{(n+1)(n+2)} \right] = \delta. \tag{14}$$

Since $n \equiv n(t)$ these become:

$$2 \frac{d\delta}{dt} - \frac{(2n+1)\delta}{n(n+1)} \frac{dn}{dt} = \frac{n(n+1)}{\delta}, \quad 3 \frac{d\delta}{dt} - \frac{(2n+3)\delta}{(n+1)(n+2)} \frac{dn}{dt} = \frac{(n+1)(n+2)}{\delta}, \tag{15}$$

or

$$\delta \frac{d\delta}{dt} = \frac{2(n+1)(3n^2+6n+2)}{2n^2+9n+6}, \quad \delta^2 \frac{dn}{dt} = \frac{n(n+1)^2(n+2)(4-n)}{2n^2+9n+6}. \tag{16}$$

From the second equation we see that constant solutions for n exist for $n = -2, n = -1, n = 0, n = 4$ and then the first gives $\delta = \sqrt{\alpha t}$ where:

$$\alpha = 2 \frac{(n+1)(3n^2+6n+2)}{2n^2+9n+6}. \tag{17}$$

We have already restricted $n > 1$ which means we must use $n = 4$, and then from (17) it follows that $\alpha = 20$. It was shown in [18] that for this problem the error is minimised by choosing $n \approx 3.584$ and $n \approx 3.822$ for HBIM and RIM respectively. So the value $n = 4$ is the closest integer value (although it is not always true that the CIM gives an integer value for n). From (14), assuming n is constant we can integrate to give $\alpha = n(n+1)$ and $\alpha = 2(n+1)(n+2)/3$ for the HBIM and RIM respectively. Obviously equating these two expressions leads to the CIM value $n = 4$.

It is interesting to note that $n = 4$ was also used in [20], there chosen after examining an expansion of the error function solution (7) in the pre-ablation stage. However, it is not clear that $n = 4$ will be a good choice in the ablation stage, which will be shown below by applying the CIM once ablation has started. One further thing to note is that for different problems n is not always constant. This depends on the type of boundary condition specified at $x = 0$. For example, if the cooling condition $T_x = T - 1$ is applied at $x = 0$ then the analogous equations to (14) would have to be solved to determine both $\delta(t)$ and $n(t)$, see [16].

Fig. 2(a) shows a comparison of three approximate solutions with the exact solution (4). This is plotted as a solid line, the CIM as a dashed line, the HBIM with a minimised error ($n \approx 3.584$) as a dash-dotted line and a standard HBIM with $n = 2$ as a dotted line. The minimised RIM solution is difficult to distinguish from the minimised HBIM and combined methods and is

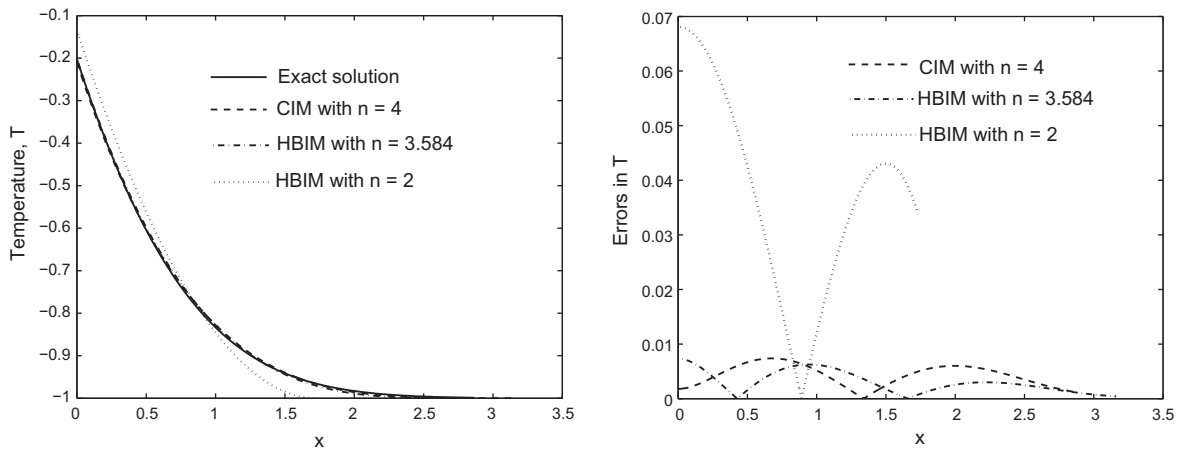


Fig. 2. Comparison of results for the problem (4), at $t = 0.5$, with the CIM (hence $n = 4$), minimised HBIM (with $n = 3.5848$) and the classical HBIM (with $n = 2$). The left plot shows the temperature T and the right plot shows the corresponding absolute errors.

therefore not plotted. This is an example where n is constant and consequently the CIM and the error minimisation method are difficult to distinguish from each other and, more importantly, from the exact solution. The classical method, with $n = 2$, is clearly the worst approximation. To make the difference between the curves clearer, Fig. 2(b) plots the absolute error, when compared to the exact solution. The standard approximation, with $n = 2$ leads to a maximum error of around 7% whereas the combined and minimised methods both have errors below 1%.

Mitchell and Myers [16] also showed that a more accurate solution could be obtained from including an extra logarithmic term in the approximating profile. The motivation for this profile came from studying the thermal problem with a time-dependent boundary condition specified at $x = 0$, namely $T(0, t) = f(t)$. It is well known that when applying a time-dependent boundary condition, heat balance methods are only possible for small time or very limited (and not physically realistic) forms of $f(t)$, see [14,13,7,11]. For example, if $f(t) = 1 - t$ then there comes a time when the internal temperature is greater than that at the boundary, i.e., there is a maximum at $x = h(t)$ where $0 < h < \delta$. The standard polynomial profile is not appropriate since it does not allow a turning point at $x \neq \delta$.

The profile was of the form:

$$T = -1 + \left[a(t) + b(t) \ln \left(1 - \frac{x}{\delta} \right) \right] \left(1 - \frac{x}{\delta} \right)^n. \tag{18}$$

A further condition is required to determine the extra coefficient: differentiating $T_x = -1$ with respect to t leads to $T_{xxx} = 0$ at $x = 0$. Using this profile leads to much higher values for n ($n \approx 7.515$ rather than $n = 4$) and about a factor 10 decrease in the maximum error.

Despite the fact that the logarithmic profile gives a better approximation for the thermal problem, we do not consider it further in this paper. One reason is that it significantly complicates the analysis, thus detracting from the simplicity of the method. In addition, it turns out that the logarithmic profile is not always more accurate than the polynomial profile for Stefan problems, and the ablation problem predominantly involves solving a Stefan problem since the pre-ablation stage takes place over a small time interval.

4. Ablation stage

Stage 1 ends at $t = t_1$ where t_1 satisfies $T(0, t_1) = 0$. From (7) the exact value of t_1 is found to be $t_1 = \pi/4$. For the CIM, setting $x = 0$ in (8) leads to $\delta(t_1) = n = 4$ and then it follows that $t_1 = n/(n + 1) = 0.8$ giving a 0.12% error. The standard HBIM (with $n = 4$) gives a 0.94% error whilst the minimised HBIM (with $n = 3.584$) gives a 0.03% error. Although this is much smaller we shall see below that applying the minimised CIM in the ablation stage is very unsatisfactory.

We now consider the problem (5) and (6) for $t > t_1$. The polynomial profile is given by

$$T = -1 + \left(\frac{\delta - x}{\delta - s} \right)^p, \tag{19}$$

where $p > 1$ to ensure $T_x = 0$ at $x = \delta$. This satisfies conditions $T(s, t) = 0$, $T(\delta, t) = -1$ and $T_x(\delta, t) = 0$. To determine an initial condition for p we assume that the profiles (19) and (8) must match at $t = t_1$; since $s(t_1) = 0$ this implies that $p(t_1) = n = 4$.

Integrating the heat equation over $[s, \delta]$ and using boundary conditions $T(s, t) = 0$, $T(\delta, t) = -1$ and $T_x(\delta, t) = 0$ leads to the HBIM formulation:

$$\frac{d}{dt} \int_s^\delta T dx + \frac{d\delta}{dt} = -\frac{\partial T}{\partial x}(s, t). \tag{20}$$

There is more freedom in the RIM formulation. Noting that $T_x(\delta, t) = 0$ we choose to first integrate over $[x, \delta]$. Since $T_x = 0$ at $x = \delta$ the first integral may be written as:

$$\int_x^\delta T_t(\xi, t) d\xi = -T_x(x, t). \tag{21}$$

The second integration is between $[s, \delta]$ and gives:

$$\int_s^\delta \int_x^\delta T_t(\xi, t) d\xi dx = 1, \tag{22}$$

after applying $T(\delta, t) = -1, T(s, t) = 0$. Changing the order of integration and switching the dummy variable to x results in:

$$\int_s^\delta (x - s)T_t(x, t) dx = 1, \tag{23}$$

and applying Leibniz's rule leads to:

$$\frac{d}{dt} \int_s^\delta xT dx - s \frac{d}{dt} \int_s^\delta T dx + (\delta - s) \frac{d\delta}{dt} = 1. \tag{24}$$

To obtain the standard RIM we replace the second integral in (24) using (20), resulting in:

$$\frac{d}{dt} \int_s^\delta xT dx + \delta \frac{d\delta}{dt} = 1 - s \frac{\partial T}{\partial x}(s, t). \tag{25}$$

It turns out to be more convenient to work with this formulation but using either (20), (24) or (20), (25) leads to the same results (since the CIM combines both the HBIM and RIM formulations).

Substituting the profile (19) into equations (20), (24) gives:

$$p \frac{ds}{dt} + \frac{d\delta}{dt} - \frac{\delta - s}{p + 1} \frac{dp}{dt} = \frac{p(p + 1)}{\delta - s}, \tag{26}$$

$$p \frac{ds}{dt} + 2 \frac{d\delta}{dt} - \frac{(2p + 3)(\delta - s)}{(p + 1)(p + 2)} \frac{dp}{dt} = \frac{(p + 1)(p + 2)}{\delta - s}. \tag{27}$$

Then the Stefan condition in (6) becomes:

$$\beta \frac{ds}{dt} = 1 - \frac{p}{\delta - s}. \tag{28}$$

Since $p(t_1) = \delta(t_1) = 4$ we see that $s_t = 0$ at $t = t_1$. This agrees with the analysis in the Appendix A where we can show analytically that $s \sim (t - t_1)^{3/2}$ as $t \rightarrow t_1^+$. Hence we can deduce that $p(t_1) = \delta(t_1)$ straight from (28), provided we assume $s_t = 0$.

The ODEs in (26) and (27) can be rearranged to give:

$$-p \frac{ds}{dt} + \frac{\delta - s}{(p + 1)(p + 2)} \frac{dp}{dt} = \frac{(p + 1)(2 - p)}{\delta - s}, \tag{29}$$

$$-p(p + 1) \frac{ds}{dt} + \frac{d\delta}{dt} = \frac{(p + 1)(-p^2 + p + 4)}{\delta - s}. \tag{30}$$

Using the fact that $s = s_t = 0$ at $t = t_1$ these reduce to:

$$\frac{\delta - s}{(p + 1)(p + 2)} \frac{dp}{dt} = \frac{(p + 1)(2 - p)}{\delta}, \quad \frac{d\delta}{dt} = \frac{(p + 1)(-p^2 + p + 4)}{\delta}. \tag{31}$$

Eq. (31b) shows that $\delta_t = -10 < 0$ as $t \rightarrow t_1^+$, whereas the pre-ablation stage gives $\delta_t = 5/2$ as $t \rightarrow t_1^-$. It is not in fact necessary for the δ_t 's to be equal at $t = t_1$ because the boundary conditions change here. If we were to apply the HBIM to both stages, as considered by Mitchell and Myers [20,8], then the pre-ablation stage gives $\delta_t = (n + 1)/2$ as $t \rightarrow t_1^-$. Once ablation starts we find that $\delta_t = n + 1$ as $t \rightarrow t_1^+$ (after examining (26) with $s = s_t = 0, p = n$ and $p_t = 0$). Hence the HBIM also has this feature. What is unexpected, and perhaps not physically realistic, is that the CIM forces $\delta_t < 0$ initially after the switch. However, despite this unfortunate feature, the results using the CIM are much better than using the standard or minimised HBIM, as we demonstrate below.

Before giving results we comment that the minimised error method has not previously been applied to the ablation problem. As mentioned in Section 2, when the exponent p is time-dependent the integral to be minimised involves time derivatives in p which cannot be evaluated. Myers [18] overcomes this difficulty by choosing the exponent to minimise the error at $t = 0$, after showing that this is where the greatest error occurs. If we were to do the same here, i.e., by choosing p to minimise the error at $t = t_1$, then it follows that $p = 2.395$. However, for this problem it is not clear that the greatest error will occur at $t = t_1$, and so to be consistent with the pre-ablation stage we take $p = n = 3.584$ for the minimised HBIM. In addition we also include results for the standard HBIM with $p = 2$, as for the pre-ablation stage.

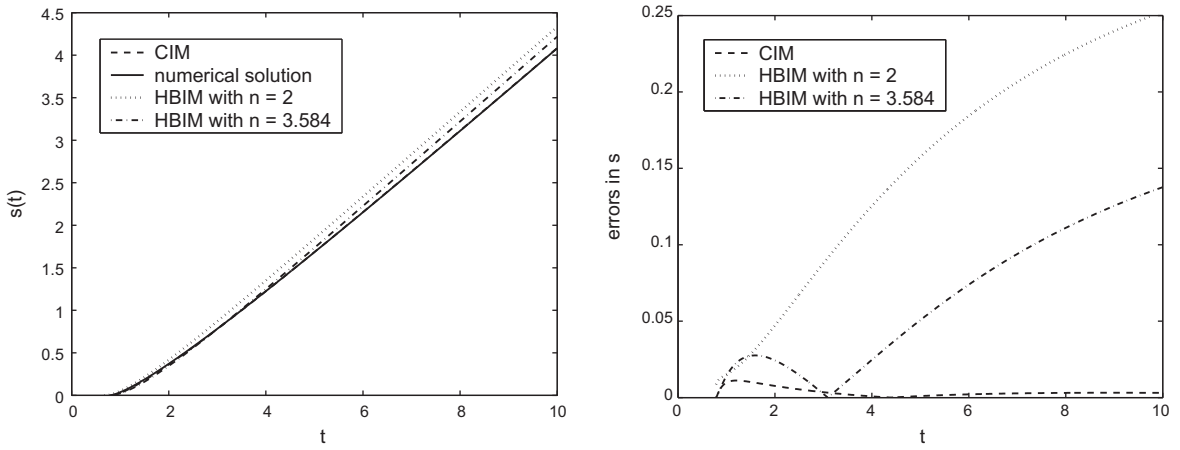


Fig. 3. Plots of $s(t)$ against t for $\beta = 1$ (left) and absolute errors compared with a numerical solution (right).

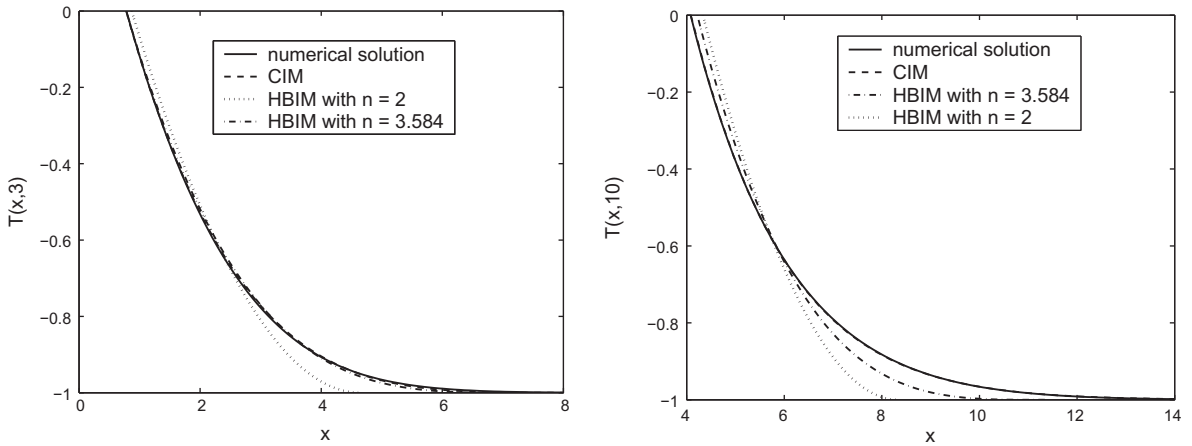


Fig. 4. Plots of T against x for $\beta = 1$ at $t = 3$ (left) and $t = 10$ (right).

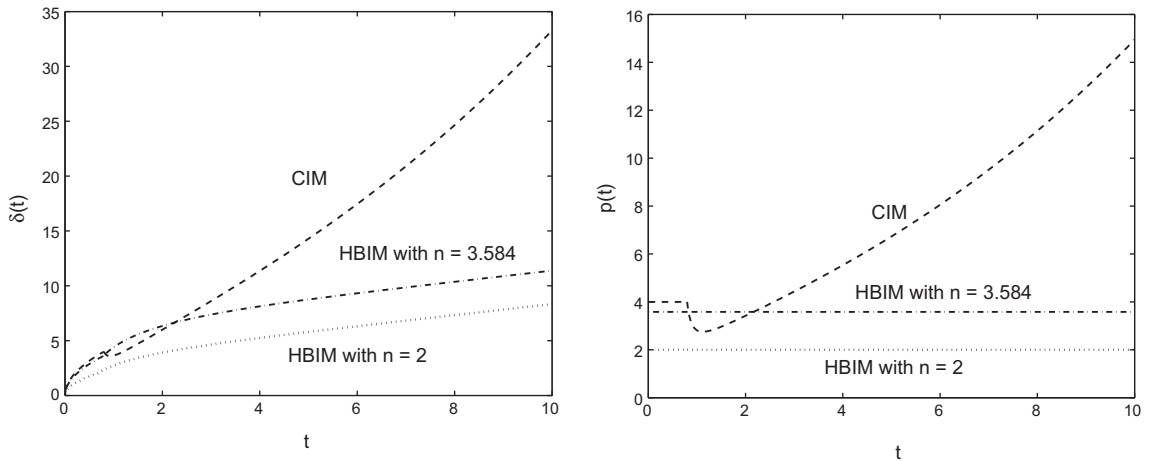


Fig. 5. Plots of $\delta(t)$ against t (left) and $p(t)$ against t (right), for $\beta = 1$.

In Fig. 3(a) we show results for the ablation depth s , with $\beta = 1$. The CIM, standard HBIM and minimised HBIM are compared with a second order accurate numerical solution [13]. It is clear that the CIM is far more accurate, Fig. 3(b) shows that the maximum error is $\approx 8 \times 10^{-3}$, and it is decreasing with time unlike the other methods. Indeed, in Fig. 3(a) it is hard to distinguish the CIM from the numerical solution. Note that the standard HBIM gives extremely bad results.

Fig. 4 plots the temperature T against x for two different values of t . Fig. 4(b), where $t = 10$, really highlights the benefit of using the CIM over both the standard and minimised HBIM for larger times. Fig. 4(a) shows $t = 3$; this value was chosen because at that time the error in s is actually smaller for the minimised HBIM than the CIM (as can be seen in the right plot in Fig. 3). Here the minimised HBIM temperature profile is closer to the CIM profile but is still less accurate. In Fig. 5(a) we show the heat penetration depth δ and in Fig. 5(b) the exponents n, p for the pre-ablation and ablation stages respectively. As predicted above, $\delta_t < 0$ just after ablation starts but this happens over a very small timescale before increasing again. In fact, even for $t \in [t_1, 0.9]$, which is when $\delta_t < 0$ (as can be seen in Fig. 5(a)), the CIM is still more accurate than the minimised HBIM. From Fig. 5(b) it is clear why the results for the minimised HBIM and CIM are more comparable for smaller times: the exponents here for each are quite similar. However, as time increases we see that p increases dramatically which accounts for the big difference in the temperature plots. Previous studies have also noticed that using larger exponents in the ablation stage leads to more accurate results, for example Braga et al. [14] use $p = 7$, although no justification is given for this choice. Mitchell and Myers [20] noted that if a constant exponent is used then it is physically unrealistic to have the exponent in the ablation stage larger than in the pre-ablation stage, since this leads to $s_t < 0$ just after the switch. Allowing p to vary means we do not have to choose the exponent in an arbitrary way, and we are not restricted to using a value which is less than that used in the pre-ablation stage.

Since the value of β can vary widely depending on the type of material, in Figs. 6 and 7 we show results for $\beta = 5$ and $\beta = 0.2$, respectively. Again the CIM is much more accurate and so for the ablation problem varying β does not affect which method is most accurate, unlike classic Stefan problems [8,17]. However, like the classic Stefan problem, the standard and minimised HBIM do break down for small β .

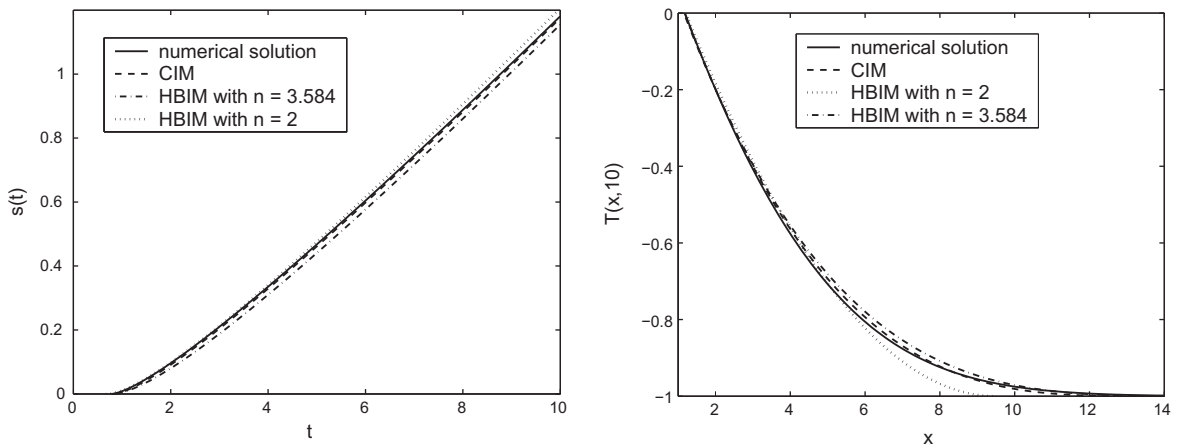


Fig. 6. Plots of $s(t)$ against t (left) and T against x (right), at $t = 10$ with $\beta = 5$.

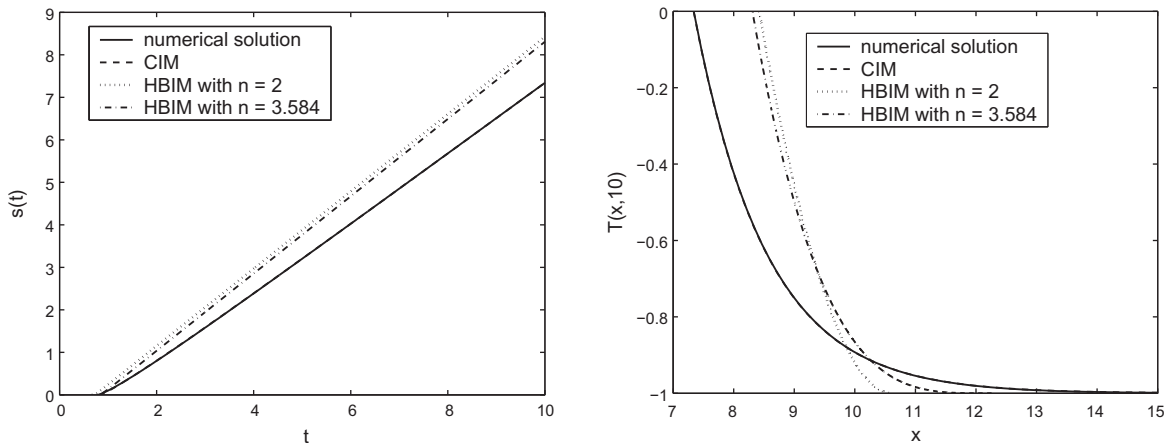


Fig. 7. Plots of $s(t)$ against t (left) and T against x (right), at $t = 10$ with $\beta = 0.2$.

5. Conclusion

In this paper the CIM has been applied to the process of one-dimensional semi-infinite ablation. Although various authors have already applied heat balance methods to this problem, the choice of approximating function has always been in debate, and this is key to providing an accurate solution. All previous work has used a constant exponent in the ablation stage: either ensuring $s_t \geq 0$ just after the switch and so restricting $p \leq n$, which is only accurate for small times [20], or violating this condition to enable larger values of p to be used [14]. For the CIM the exponent p initially decreases in the ablation stage, and thus ensuring $s_t \geq 0$, but then over time increases to a value larger than n (as seen in the right plot in Fig. 5). This is not a problem, however, because there is no danger in s_t becoming negative at this time.

Another key advantage of the CIM is that it removes the choice between whether to use the HBIM or RIM formulations. Both methods can be shown to be the most accurate for certain problems and parameter values [8], and this ambiguity is one reason why these approximate methods have been criticised. The minimisation method of Myers [18,19] is generally more accurate than the CIM when the exponent is constant, but it does not allow the exponent to be time-dependent and using a constant value affects its accuracy. With the present method there is no such problem as p simply satisfies a first order differential equation and the results show a significant increase in accuracy. Although the CIM is slightly more complicated than the standard heat balance integral method, namely requiring one extra ODE to be solved, it is clearly worth the effort given this dramatic improvement.

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Appendix A

Let us consider the limit $t \rightarrow t_1^+$. We begin by making the change variables:

$$\xi = x - s, \quad T(x, t) = F(\xi, t).$$

Then the system (5) and (6) becomes:

$$\frac{\partial^2 F}{\partial \xi^2} = \frac{\partial F}{\partial \tau} - \dot{s} \frac{\partial F}{\partial \xi}, \quad 0 < \xi < \infty, \quad (32)$$

$$F = 0, \quad \beta \frac{ds}{dt} = 1 + \frac{\partial F}{\partial \xi}, \quad \text{at } \xi = 0, \quad (33)$$

$$\frac{\partial F}{\partial \xi} \rightarrow 0, \quad \text{as } \xi \rightarrow \infty, \quad (34)$$

$$s = 0, \quad \text{at } t = t_1. \quad (35)$$

Suppose that s has the form:

$$s \sim \lambda(t - t_1)^\alpha + \vartheta(t - t_1)^\alpha. \quad (36)$$

For convenience we also set $\tau = t - t_1$ and then using the form of s in (36) the PDE (32) becomes:

$$F_\tau - \lambda\alpha\tau^{\alpha-1}F_\xi = F_{\xi\xi}, \quad 0 < \xi < \infty, \quad \tau > 0. \quad (37)$$

Using Fourier Transforms we can find the solution, as $\tau \rightarrow 0$, of the related problem defined on $-\infty < \xi < \infty$ subject to $F_\xi \rightarrow 0$ as $\xi \rightarrow \pm\infty$ and $F(\xi, 0) = F_0(\xi)$. It follows that:

$$F(\xi, \tau) = \frac{1}{2} \sqrt{\frac{1}{\pi\tau}} \int_{-\infty}^{\infty} F_0(\xi') \exp\left(-\frac{(\xi + \lambda\tau^\alpha - \xi')^2}{4\tau}\right) d\xi'. \quad (38)$$

Following Carslaw and Jaeger [22], we can use (38) to find the solution of (37) subject to $F(0, \tau) = 0$, $F_\xi \rightarrow 0$ as $\xi \rightarrow \infty$ and $F(\xi, 0) = F_0(\xi)$. Thus:

$$F(\xi, \tau) = \frac{1}{2} \sqrt{\frac{1}{\pi\tau}} \int_0^\infty F_0(\xi') \left\{ \exp\left(-\frac{(\xi + \lambda\tau^\alpha - \xi')^2}{4\tau}\right) - \exp\left(-\frac{(\xi - \lambda\tau^\alpha + \xi')^2}{4\tau}\right) \right\} d\xi'. \quad (39)$$

We wish to substitute this expression into (33) to determine s . It follows from (39) that:

$$(F_\xi)_{\xi=0} = \frac{1}{2} \sqrt{\frac{1}{\pi\tau^3}} \int_0^\infty F_0(\xi') (\xi' - \lambda\tau^\alpha) \exp\left(-\frac{(\xi' - \lambda\tau^\alpha)^2}{4\tau}\right) d\xi' = 2 \sqrt{\frac{1}{\pi\tau}} \int_{-\lambda\tau^\alpha/(2\sqrt{\tau})}^\infty F_0(2\phi\sqrt{\tau} + \lambda\tau^\alpha) \phi e^{-\phi^2} d\phi, \quad (40)$$

where $\phi = (\xi^t - \lambda\tau^\alpha) / [2\sqrt{\tau}]$. We now expand F_0 about 0:

$$F_0(2\phi\sqrt{\tau} + \lambda\tau^\alpha) \approx F_0(0) + (2\phi\sqrt{\tau} + \lambda\tau^\alpha)F_{0\xi}(0) + \frac{1}{2}(2\phi\sqrt{\tau} + \lambda\tau^\alpha)^2 F_{0\xi\xi}(0) + \dots,$$

and assuming that $F_0(0) = 0$, which we will see is true below, means that (40) becomes:

$$(F_\xi)_{\xi=0} = 2\sqrt{\frac{1}{\pi\tau}} F_{0\xi}(0) \int_{-\lambda\tau^\alpha/(2\sqrt{\tau})}^{\infty} (2\phi\sqrt{\tau} + \lambda\tau^\alpha)\phi e^{-\phi^2} d\phi + \sqrt{\frac{1}{\pi\tau}} F_{0\xi\xi}(0) \int_{-\lambda\tau^\alpha/(2\sqrt{\tau})}^{\infty} (2\phi\sqrt{\tau} + \lambda\tau^\alpha)^2 \phi e^{-\phi^2} d\phi + \dots \quad (41)$$

Calculating these integrals leads to the expression:

$$(F_\xi)_{\xi=0} = F_{0\xi}(0)\operatorname{erfc}\left(-\frac{\lambda\tau^\alpha}{2\sqrt{\tau}}\right) + F_{0\xi\xi}(0)\left[2\sqrt{\frac{\tau}{\pi}}\left(1 + \frac{\lambda^2\tau^{2\alpha-1}}{4}\right)\exp\left(-\frac{\lambda^2\tau^{2\alpha-1}}{4}\right) + \lambda\tau^\alpha\operatorname{erfc}\left(-\frac{\lambda\tau^\alpha}{2\sqrt{\tau}}\right)\right] + \dots \quad (42)$$

If we expand (42) for small τ we obtain:

$$(F_\xi)_{\xi=0} \approx F_{0\xi}(0)\left(1 + \frac{\lambda}{2}\tau^{\alpha-1/2} + \dots\right) + F_{0\xi\xi}(0)2\sqrt{\frac{\tau}{\pi}} + \dots \quad (43)$$

The Stefan condition in (33) now becomes (replacing t by τ and using s from (36)):

$$\beta\lambda\alpha\tau^{\alpha-1} = 1 + F_{0\xi}(0)\left(1 + \frac{\lambda}{2}\tau^{\alpha-1/2} + \dots\right) + F_{0\xi\xi}(0)2\sqrt{\frac{\tau}{\pi}} + \dots \quad (44)$$

In particular, if we use:

$$F_0(\xi) = -1 + 2\sqrt{\frac{\tau_1}{\pi}}\exp\left(-\frac{\xi^2}{4t_1}\right) - \xi\operatorname{erfc}\left(\frac{\xi}{2\sqrt{t_1}}\right), \quad (45)$$

which comes from (7), then:

$$F_{0\xi}(0) = -1, \quad \& \quad F_{0\xi\xi}(0) = \frac{1}{\sqrt{\pi t_1}},$$

and $F_0(0) = 0$, which justifies this assumption above. Finally (44) reduces to:

$$\beta\lambda\alpha\tau^{\alpha-1} = -\frac{\lambda}{2}\tau^{\alpha-1/2} + \frac{2}{\pi}\sqrt{\frac{\tau}{t_1}} + \dots \quad (46)$$

To obtain the correct balance we must set $\alpha = 3/2$ and then:

$$\lambda = \frac{4}{3\beta\pi\sqrt{t_1}} = \frac{8}{3\beta\pi^{3/2}}. \quad (47)$$

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