Finite element approximation of a nonlinear heat conduction problem in anisotropic media

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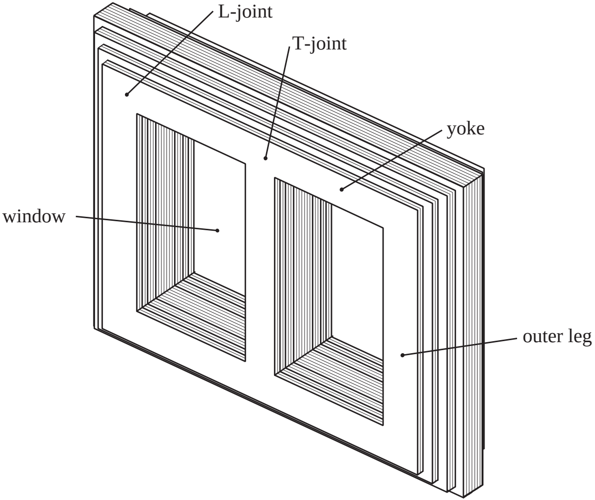
1. Motivation

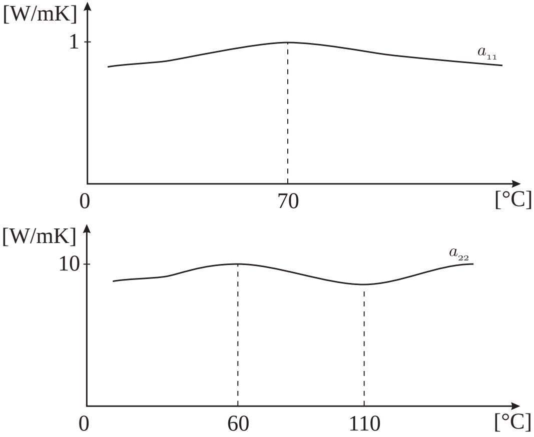
Air-cooled high-voltage transformer

1. Motivation



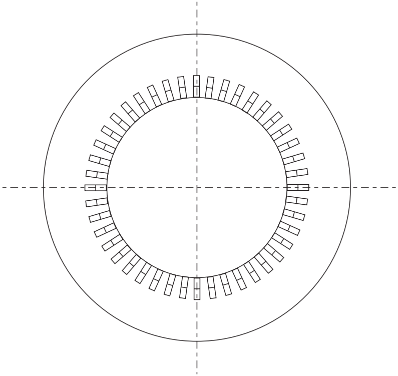
Magnetic circuit of a transformer with iron sheets — orthotropic homogeneous nonlinear material, height = 5 m, power 400 MW.

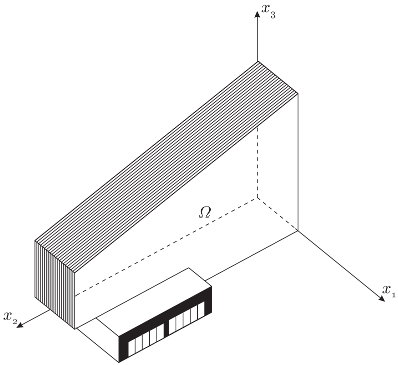
1. Motivation



The associated 3×3 matrix $A = A(u)$ of heat conductivities is diagonal and such that $a_{11} \neq a_{22} = a_{33}$. The Kirchhoff transformation cannot be applied in the case of anisotropic nonlinear media.

1. Motivation





Stator of a rotary machine — anisotropic nonhomogeneous and nonlinear material, $a_{ii}(Cu) = 332$ W/(mK), $a_{11}(Is) = 0.2$ W/(mK), $a_{22}(Is) = 0.5$ W/(mK), $R = 1$ m, power 50 MW.

1. Motivation

A challenge for numerical mathematics:

1. The main goal of the lecture is to show why it is important to deal with Hilbert spaces, imbedding theorems, weak convergence, monotone operators, compact sets, etc., in solving real-life technical problems.

2. We also hope that it will be clear why it is important to deal with material anisotropy, inhomogeneities, various nonlinearities and complicated geometry of electrical machines.

3. Such facts do not occur when solving academic examples with the Laplace operator on a square or circle.

2. Classical formulation

The problem of a stationary heat conduction in nonhomogeneous, anisotropic and nonlinear media consists of finding $u \in C^1(\overline{\Omega}) \cap C^2(\Omega)$ such that

$$(P) \qquad -\mathrm{div}(A(x,u)\,\mathrm{grad}\, u) = f \quad \text{in} \quad \Omega,$$

$$\alpha u + n^T A(s,u)\,\mathrm{grad}\, u = g \quad \text{on} \quad \partial\Omega,$$

where $\Omega \subset R^d$, $d \in \{1,2,3\}$, is a bounded domain with Lipschitz continuous boundary $\partial\Omega$, n is the outward unit normal to $\partial\Omega$, u is the temperature, $f \in L^2(\Omega)$ is the density of volume heat sources, $g \in L^2(\partial\Omega)$ is the density of surface heat sources, $\alpha \in L^\infty(\partial\Omega)$ is the heat transfer coefficient such that

$$(C) \qquad \alpha(s) \geq C > 0 \quad \forall s \in \Gamma,$$

where C is a constant, and $\Gamma \subset \partial\Omega$ has a positive measure.

2. Classical formulation

Assume further that

$$(A) \qquad A = (A_{ij})_{i,j=1}^{d} \in (L^{\infty}(\Omega))^{d\times d}$$

is a matrix function of heat conductivities such that there exist positive constants C_E and C_L for which

$$(E) \qquad \eta^T A(x,\xi)\eta \geq C_E \|\eta\|^2 \quad \forall x \in \Omega \quad \forall \xi \in R^1 \quad \forall \eta \in R^d$$

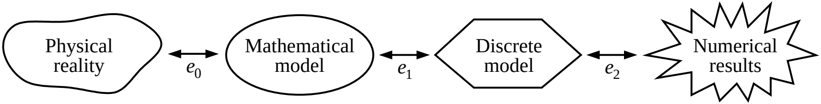
and

$$(L) \qquad |A_{ij}(x,\zeta) - A_{ij}(x,\xi)| \leq C_L |\zeta - \xi| \;\forall x \in \Omega \;\; \forall \zeta, \xi \in R^1.$$

Theorem. *Let (A), (C), (E), (L) hold and let A be a diagonal matrix such that A_{ii} are continuously differentiable on $\overline{\Omega} \times R^1$ in all arguments. Then there exists at most one solution of the classical problem.*

See I. Hlaváček, M. Křížek, Stab. Appl. Anal. Contin. Media **3** (1993), 85–97.

2. Classical formulation



General computational scheme

e_0 – modelling error

e_1 – discretization error

e_2 – computational error (iteration and rounding errors)

$$|e_0| \le |e_1| + |e_2| + |e_3|$$

where $e_3 = e_0 + e_1 + e_2$ is the total error.

3. Weak formulation

From now on, we shall assume that conditions (C), (E), and (L) hold almost everywhere on Γ and Ω. Set

$$a(y; w, v) = (A(y)\mathrm{grad}\, w, \mathrm{grad}\, v)_{0,\Omega} + \langle \alpha w, v\rangle_{0,\partial\Omega},$$

$$F(v) = (f, v)_{0,\Omega} + \langle g, v\rangle_{0,\partial\Omega},$$

where $v, w, y \in V = H^1(\Omega)$, $A(y) = A(\cdot, y)$, and $\langle ., .\rangle_{0,\partial\Omega}$ stands for the usual scalar product in $L^2(\partial\Omega)$.

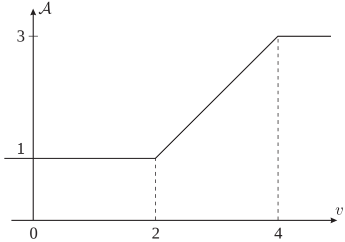
A function $u \in V$ is said to be a *weak solution* of the problem (P) if

$$a(u; u, v) = F(v) \quad \forall v \in V.$$

Using Green's theorem, we can check that the classical solution of (P) is also the weak solution.

3. Weak formulation

To prove the existence of a weak solution $u \in V$ we cannot apply the Minty–Browder theorem for monotone operators, since our problem does not lead to a monotone operator, in general. To see this we put $d = 1$, $\Omega = (0, 1)$, $\alpha = 1/50$ on $\Gamma = \partial\Omega$, and let A be given as follows



Define a nonlinear operator $\mathbb{A} : V \to V$ by the Riesz reprezentation theorem

$$\langle \mathbb{A}w, v \rangle = a(w; w, v), \quad v, w \in V,$$

where $\langle \cdot\,, \cdot \rangle$ is the $H^1(\Omega)$-scalar product.

3. Weak formulation

We see that the functions $v(x) = 2x$ and $w(x) = x + 4$ violate the monotonicity condition for the operator $\mathbb{A}$, since

$$\langle \mathbb{A}v - \mathbb{A}w, v - w \rangle = a(v; v, v - w) - a(w; w, v - w)$$

$$= \int_0^1 (A(v)v' - A(w)w')(v' - w')\, dx + \frac{1}{50} \int_\Gamma (v - w)^2\, ds$$

$$= \int_0^1 (1 \times 2 - 3 \times 1)(2 - 1)\, dx + \frac{1}{50} \left((v(1) - w(1))^2 + (v(0) - w(0))^2 \right)$$

$$= -1 + \frac{1}{2} < 0,$$

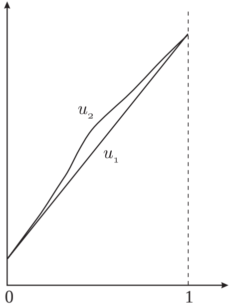
where v' denotes the derivative with respect to x.

Th. *Let $J : V \to R^1$ be Gâteaux differentiable and $\langle \mathbb{A}v, w \rangle := J'(v; w)$. Then J is (strictly) convex iff $\mathbb{A}$ is (strictly) monotone.*

Unfortunately, problem (P) is nonmonotone and is also nonpotential.

4. Nonuniqueness and uniqueness

$$\text{Set} \quad A(x,\xi) = \frac{1}{u_i'(x)} \quad \text{for } x \in [0,1],\ \xi = u_i(x),\ i = 1,2.$$



Then by Tietze's extension theorem (see [Rudin]) there exists a continuous extension (still denoted by A) so that $A(\cdot\,,\cdot) : \Omega \times R^1 \to R^1$ is bounded and (A), (C), and (E) hold. We see that

$$-(A(x,u_i)u_i')' = 0 \quad \text{for } i = 1,2,$$

i.e., u_1 and u_2 are solutions of (P) with nonhomogeneous boundary conditions and $f = 0$.

However, in this case it is not difficult to check that A is not Lipschitz continuous (with respect to the second variable) near those points, where u_1 and u_2 bifurcate. The condition (L) is essential:

Theorem. *Let (A), (C), (E), (L) hold and let $u_1, u_2 \in V$ be two weak solutions of problem (P). Then $u_1 = u_2$ a.e. in Ω.*

For the proof see I. Hlaváček, M. Křížek, J. Malý: *On Galerkin approximations of a quasilinear nonpotential elliptic problem of a nonmonotone type.* J. Math. Anal. Appl. **184** (1994), 168–189.

Remark. Another uniqueness theorems for nonlinear elliptic problems with Dirichlet boundary conditions are given in Boccardo, Gallouët, Murat (1992) and Jensen (1988).
If a nonlinear elliptic equation is not in the divergence form, there exist examples of nonunique solutions, see, e.g., Gilbarg, Trudinger (1977) or Meyers (1963). For semiconductor equations see Markowich et al. (1986, 1990).

4. Nonuniqueness and uniqueness

Assume that there exists a positive constant C_H such that

$$(H) \qquad |A_{ij}(x,\zeta) - A_{ij}(x,\xi)| \le C_H|\zeta - \xi|^e \quad \forall x \in \Omega \ \forall \zeta, \xi \in R^1,$$

where $e \in \left[\frac{1}{2}, 1\right]$ is a given Hölder exponent, i.e., $A(\cdot,\cdot)$ is e-Hölder continuous with respect to the last variable.

Theorem. *Let (C), (E), (H) hold and let $u_1, u_2 \in V$ be two weak solutions of problem (P). Then $u_1 = u_2$ a.e. in Ω.*

For the proof see M. Křížek: *The uniqueness of the solution of a nonlinear heat conduction problem under Hölder's continuity condition.* Appl. Math. Lett. **103** (2020), Article 106214, 1–6.
The proof involves also a nonlinear dependence of f and g on u, and mixed Dirichlet-Newton boundary conditions.

5. Comparison principle

Comparison and maximum principles are important features of second order equations that distinguish them from higher order equations and systems of equations.

Theorem. *Let (A), (C), (E), (L) hold and let $u_1, u_2 \in V$ be two weak solutions of problem (P) corresponding to $f_1, f_2 \in L^2(\Omega)$ and $g_1, g_2 \in L^2(\partial\Omega)$, respectively. Assume that*

$$f_1 \geq f_2 \quad \text{a.e.} \quad \text{in} \quad \Omega$$

and

$$g_1 \geq g_2 \quad \text{a.e.} \quad \text{on} \quad \partial\Omega.$$

Then $u_1 \geq u_2$ a.e. in Ω.

Note that the comparison principle immediately implies the uniqueness of the weak solution. The comparison principle also yields a natural assertion: Any rise of the density of heat sources always causes that the temperature will not decrease in any point. This confirms that the nonlinear mathematical model (P) of stationary heat conduction has reasonable properties.

Theorem. *Let (A), (C), (E), (L) hold and let $V_h \subset C(\overline{\Omega})$ be a nonempty finite-dimensional subspace. Then there is a Galerkin solution $u_h \in V_h$ such that*

$$(G) \qquad a(u_h; u_h, v_h) = F(v_h) \quad \forall v_h \in V_h.$$

See Hlaváček, Křížek, Malý (1994). The proof is based on the Brouwer fixed-point theorem.

Remark. The uniqueness of u_h was recently proved by Pollock and Zhu in Numer. Math. 2018.

6. Existence of weak and discrete solutions

Let $\{V_h\}_{h\to 0}$ be a family of finite-dimensional subspaces of $H^1(\Omega) \cap C(\overline{\Omega})$ such that

$$(D) \qquad \forall v \in C^\infty(\overline{\Omega})\ \exists \{v_h\}_{h\to 0}:\ v_h \in V_h,\ \|v - v_h\|_{1,\Omega} \to 0 \text{ as } h \to 0.$$

Theorem. *Let (A), (C), (D), (E), (L) hold and let $\{u_h\}_{h\to 0}$, $u_h \in V_h$, be a sequence of Galerkin solutions satisfying (G). Then there exist a subsequence (denoted in the same way) and $u \in H^1(\Omega)$ such that*

$$(W) \qquad u_h \rightharpoonup u \quad (\textit{weakly}) \textit{ in } H^1(\Omega) \quad \textit{as} \quad h \to 0,$$

and u is a weak solution of problem (P).

See Hlaváček, Křížek, Malý (1994). The proof is based on the Eberlein-Schmulyan theorem. It can be generalized to the so-called variational crimes.

7. Convergence of finite element approximations

From the weak convergence (W) and the compactness of the imbedding operator $H^1(\Omega) \to L^2(\Omega)$ (the Rellich theorem) we can easily prove the convergence of the Galerkin solutions in the $\|.\|_{0,\Omega}$-norm. To prove even the (strong) convergence in the $\|.\|_{1,\Omega}$-norm, we shall, in addition, require that

$$(B) \qquad V_h \subset W_4^1(\Omega), \quad \|v_h\|_{1,4,\Omega} \leq C(v) \quad \forall h \in (0, h_0),$$

where $v \in C^\infty(\overline{\Omega})$, v_h satisfies (D), $h_0 > 0$, and $C(v)$ is a constant independent of h. Functions v_h can be defined as the V_h-interpolant The next theorem establishes convergence of the sequence $\{u_h\}_{h\to 0}$ without any regularity assumptions on the weak solution u.

Theorem. *Let (A), (B), (C), (D), (E), and (L) hold. Then the convergence (W) is strong, i.e.,*

$$\|u - u_h\|_{1,\Omega} \to 0 \quad \text{as } h \to 0.$$

See Hlaváček, Křížek, Malý (1994).

8. A priori error estimates

Consider the Dirichlet problem

$$(P') \qquad -\mathrm{div}(A(x,u)\,\mathrm{grad}\,u) = f \quad \text{in } \Omega,$$
$$u = 0 \quad \text{on } \partial\Omega,$$

where $f \in L^2(\Omega)$ and A is an L^∞ matrix function which is Lipschitz continuous with respect to the last variable and is uniformly positive definite, i.e., (E) holds.

The weak formulation of (P') consists of finding $u \in V$ such that

$$a(u;u,v) = F(v) \quad \forall v \in V = H_0^1(\Omega),$$

where

$$a(y;w,v) = (A(y)\,\mathrm{grad}\,w, \mathrm{grad}\,v)_{0,\Omega} \quad \text{and} \quad F(v) = (f,v)_{0,\Omega}$$

for $y, w, v \in H^1(\Omega)$.

Let the Galerkin solution u_h belong to the space

$$V_h = \{v_h \in V \mid \quad v_h|_K \in P_K \quad \forall K \in T_h\},$$

where $P_K \supset P_k(K)$ and $k \geq 1$ is an integer.

Introduce the adjoint problem: Find $\phi \in H_0^1(\Omega)$ such that

$$-\mathrm{div}(A^T(x,u)\mathrm{grad}\,\phi) + (\mathrm{grad}\,u)^T B^T(x,u)\mathrm{grad}\,\phi = \zeta,$$

where u is the weak solution of (P'), $\zeta \in L^2(\Omega)$, $B = (B_{ij})$, $B_{ij}(x,\xi) = \partial A_{ij}(x,\xi)/\partial\xi$ and, moreover, we assume that

$$(F) \qquad \|\phi\|_{2,\Omega} \leq C\|\zeta\|_{0,\Omega}.$$

Theorem. *Let $u \in H^{k+1}(\Omega)$, $k \geq 1$, be the solution of (P'), let (A), (E), (F), and (L) hold, let the derivatives $\partial A_{ij}/\partial\xi$ and $\partial^2 A_{ij}/\partial\xi^2$ be bounded and continuous on $\overline{\Omega} \times R^1$ and let $\{T_h\}_{h\to 0}$ be a regular family of triangulations. Then there exists $h_0 > 0$ such that for any $h \in (0, h_0)$ we have*

$$\|u - u_h\|_{0,\Omega} + h\|u - u_h\|_{1,\Omega} \leq Ch^{k+1},$$

where C depends on the norm $\|u\|_{k+1,\Omega}$.

9. Superconvergence by post-processing

First define a one-dimensional operator on a uniform mesh by

$$I_{2h}^{k+1}u(p_i) = u(p_i), \quad i = 1,2,3,\ k \geq 1,$$

$$\int_{\ell_j} I_{2h}^{k+1}u\ ds = \int_{\ell_j} u\ ds, \quad j = 1,2,\ k \geq 2,$$

$$\int_L I_{2h}^{k+1}uv\ ds = \int_L uv\ ds\ \forall v \in P_{k-3}(L)/P_0(L),\ L = \ell_1 \cup \ell_2,\ k \geq 3.$$

Set $\quad I_{2h}^{k+1}(x_1, x_2) := I_{2h}^{k+1}(x_1) \otimes I_{2h}^{k+1}(x_2).$

Theorem. *Let (A), (B), (C), (D), (E), and (L) hold. Then we have*

$$\|u - I_{2h}^{k+1}u_h\|_1 \leq Ch^{k+1}\|u\|_{k+2}, \quad k \geq 1,$$

$$\|u - I_{2h}^{k+1}u_h\|_0 \leq Ch^{k+2}\|u\|_{k+2}, \quad k \geq 2.$$

See L. Liu, T. Liu, M. Křížek, T. Lin, S. Zhang (2004).

10. Radiation boundary conditions

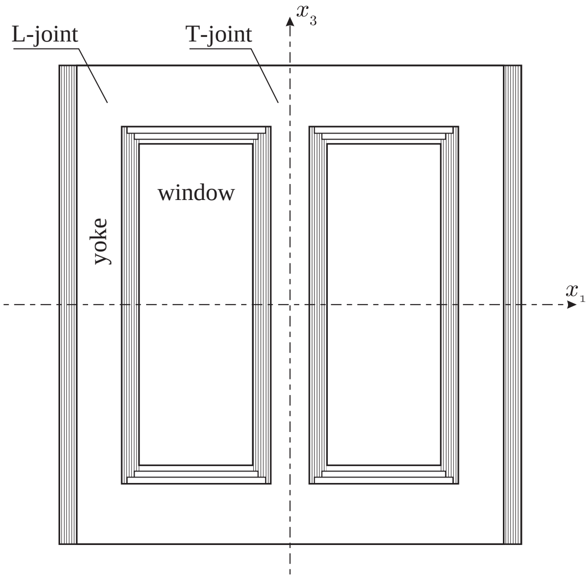
Each body loses heat energy from its surface by electromagnetic waves. This phenomenon is called radiation. Losses of energy are proportional to the fourth power of the surface temperature (the Kirchhoff law). This effect is small at room temperature. But the radiation should not be neglected when the surface temperature is high. It is represented by the nonlinear Stefan-Boltzmann boundary condition

$$\alpha u + n^T A \operatorname{grad} u + \beta u^4 = g,$$

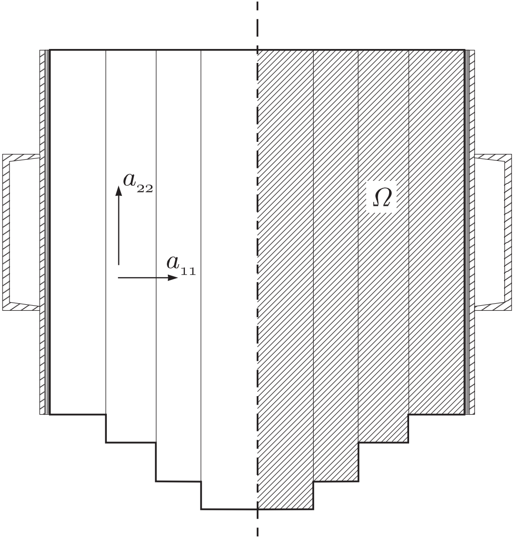
on some part of the boundary $\partial\Omega$, where $\beta = \sigma f_{\mathrm{em}}$, $\sigma = 5.669 \times 10^{-8}$ $\mathrm{Wm}^{-2}\mathrm{K}^{-4}$ is the Stefan-Boltzmann constant, $0 \le f_{\mathrm{em}} \le 1$ is the relative emissivity function.

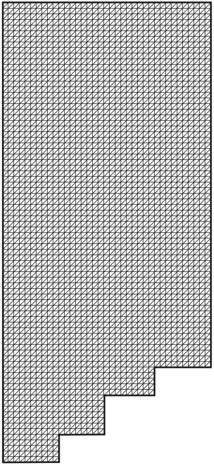
If A is independent of the solution u, the heat radiation problem can be transformed to the minimization of a nonquadratic functional over a nonempty convex set. The existence and uniqueness of the weak solution u is guaranteed.

11. Numerical example



11. Numerical example



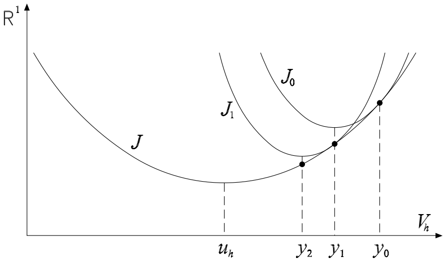


11. Numerical example

Let $A(x, u) = A(x)$ and

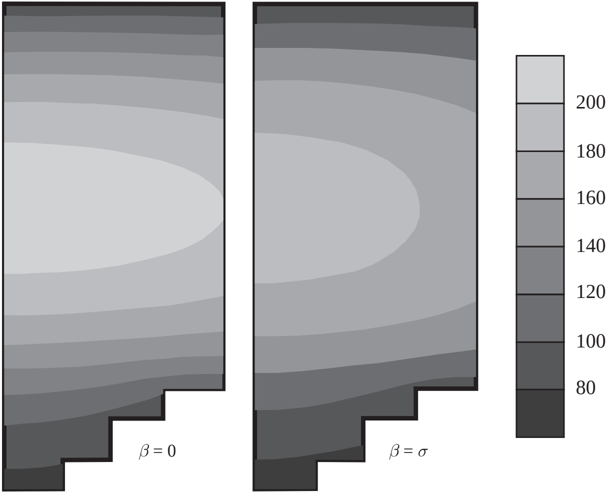
$$J(v) = \frac{1}{2}a(v, v) + \frac{1}{5}\int_{\Gamma_2} \beta v^5 ds - F(v), \quad v \in H^1(\Omega)$$

The integral is finite, since $\|v\|_{0,q,\partial\Omega} \leq C_q \|v\|_{1,2,\Omega}$. The functional J is not convex. Thus, we have restricted ourselves to $v \geq 0$.



We used Kačanov's method (= the method of secant modules = the method of freezing coefficients) over a finite element space V_h.

11. Numerical example



Here $f = 15\,000$ W/m^3, $g = 0$, $\alpha_{\mathrm{left}} = 0$, $\alpha_{\mathrm{right}} = 10$ and 100 Wm^{-1}K^{-1} otherwise.

12. Discrete maximum principle

Consider the problem

$$-\mathrm{div}(\lambda(x, u, \mathrm{grad}\, u)\mathrm{grad}\, u) = f \quad \text{ in } \Omega \subset R^3$$

with the zero Dirichlet boundary conditions, where λ is a sufficiently smooth function such that $0 < c_1 \leq \lambda(\cdot, \cdot, \cdot) \leq c_2$. The corresponding continuous maximum principle takes the from

$$f \leq 0 \quad \Longrightarrow \quad u \leq 0.$$

By M. Křížek, Qun Lin (1995) the following discrete maximum principle holds

$$f \leq 0 \quad \Longrightarrow \quad u_h \leq 0.$$

for linear tetrahedral elements. In addition the effect of numerical integration was analyzed and the DMP was proved for nonobtuse tetrahedral elements, see also J. Karátson, S. Korotov, M. Křížek (2007).

13. Final remarks

Finally, consider a stationary heat conduction problem in a bounded homogeneous and isotropic medium $\Omega \subset R^d$, $d \in \{1, 2, \dots\}$,

$$-\mathrm{div}(\lambda(u)\mathrm{grad}\, u) = f \quad \text{ in } \Omega$$

with mixed boundary conditions

$$u = 0 \quad \text{on } \Gamma_1 \neq \emptyset \quad \text{and} \quad n^\top \lambda(u)\mathrm{grad}\, u = 0 \quad \text{on } \Gamma_2,$$

where Γ_1 and Γ_2 are relatively open sets in the boundary $\partial\Omega$,

$$\overline{\Gamma}_1 \cup \overline{\Gamma}_2 = \partial\Omega, \qquad \Gamma_1 \cap \Gamma_2 = \emptyset,$$

$\lambda : R^1 \to R^1$ is a measurable bounded function such that

$$\lambda(\xi) \geq C > 0 \quad \forall \xi \in R^1,$$

i.e. $A(\cdot, u) = \lambda(u)I$, where I is the identity matrix.

13. Final remarks

This nonlinear problem can be converted by the well-known Kirchhoff transformation

$$\mathcal{K}(U) = \int_0^U \lambda(\xi)\, d\xi, \quad U \in R^1,$$

to the linear problem

$$-\Delta z = f \quad \text{in } \Omega$$

with mixed boundary conditions

$$z = 0 \quad \text{on } \Gamma_1 \quad \text{and} \quad n^\top \operatorname{grad} z = 0 \quad \text{on } \Gamma_2,$$

where $z(x) = \mathcal{K}(u(x))$. We observe that $\mathcal{K}$ is an increasing function, i.e., its inverse $\mathcal{K}^{-1}$ exists and we have

$$u(x) = \mathcal{K}^{-1}(z(x)).$$

Open problem: **Define an analogue of the Kirchhoff transformation for anisotropic material.**

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