Simplest random walks for boundary value problems

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Plan of the talk

- Introduction
- Dirichlet problem for parabolic and elliptic linear PDEs [Milstein, T 2002]
- Robin problem for parabolic and elliptic linear PDEs [Leimkuhler, Sharma,T 2022?]
- Dirichlet problem for linear PIDEs [Deligiannidis, Maurer, T, 2021]
- Conclusions

Introduction

$$L := \frac{\partial}{\partial t} + \frac{1}{2}\sum_{r=1}^{q}\sum_{i,j=1}^{d}\sigma_r^i\sigma_r^j\frac{\partial^2}{\partial x^i \partial x^j} + \sum_{i=1}^{d} b^i \frac{\partial}{\partial x^i} \tag{1}$$

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Then

$$u(t_0,x) = Ef(X_{t_0,x}(T)), \quad (4)$$

where $X_{t_0,x}(t)$, $t \geq t_0$, is the solution of the Ito SDEs

$$dX = b(t,X)dt + \sum_{r=1}^{q}\sigma_r(t,X)dw_r(t), \quad X(t_0) = x. \quad (5)$$

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Approximation:

$$u \equiv Ef(X(T)) \simeq \bar{u} \equiv Ef(X_N) \simeq \hat{u} \equiv \frac{1}{M}\sum_{m=1}^{M} f(X_N^{(m)}), \quad (6)$$

where $X_N^{(m)}$, $m = 1, \dots, M$, are independent realizations of X_N.

Numerics - weak convergence

Definition

If an approximation $\bar{X}$ is such that

$$|Ef(\bar{X}(T)) - Ef(X(T))| \leq Kh^p \tag{7}$$

for f from a class of functions with polynomial growth at infinity, then we say that the **weak order of accuracy** *of the approximation $\bar{X}$ (the method $\bar{X}$) is p.* The constant K depends on the SDE coefficients, on the function f and on T.

The *weak Euler scheme* (Milstein (1978))

$$X_{k+1} = X_k + b_k h + \sqrt{h} \sum_{r=1}^{q} \sigma_{rk} \eta_{rk}\,, \tag{8}$$

where $\eta_{rk},\ r = 1, \ldots, q,\ k = 0, \ldots, N-1$, are independent random variables taking the values $+1$ and -1 with probabilities $1/2$, also has first order of accuracy in the sense of weak approximation.
[e.g. Milstein, T.; Springer, 2004 or 2021]

Dirichlet problem

Let G be a bounded domain in $\mathbf{R}^d$ and $Q = [T_0, T) \times G \subset \mathbf{R}^{d+1}$, and $\Gamma = \bar{Q} \backslash Q$. Consider the Dirichlet problem

$$\frac{\partial u}{\partial t} + \frac{1}{2} \sum_{i,j=1}^{d} a^{ij}(t,x) \frac{\partial^2 u}{\partial x^i \partial x^j} + \sum_{i=1}^{d} b^i(t,x) \frac{\partial u}{\partial x^i} + c(t,x)u \tag{9}$$

$$+g(t,x) = 0, \ (t,x) \in Q,$$

$$u \,|_{\Gamma} = \varphi(t,x)\,. \tag{10}$$

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The probabilistic representation:

$$u(t,x) = E \left[\varphi(\tau, X_{t,x}(\tau)) Y_{t,x,1}(\tau) + Z_{t,x,1,0}(\tau) \right] , \qquad (11)$$

where $X_{t,x}(s)$, $Y_{t,x,y}(s)$, $Z_{t,x,y,z}(s)$, $s \geq t$, is the solution of the SDEs:

$$dX = (b(s,X) - \sigma(s,X)\mu(s,X))\, ds + \sigma(s,X)\, dw(s), \ \ X(t) = x, \qquad (12)$$

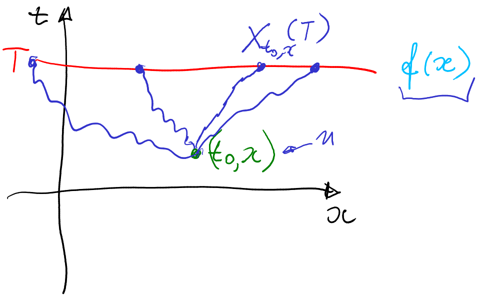
$$dY = c(s,X) Y\, ds + \mu^\intercal(s,X) Y\, dw(s), \ \ Y(t) = y, \qquad (13)$$

$$dZ = g(s,X) Y\, ds + F^\intercal(s,X) Y\, dw(s), \ \ Z(t) = z, \qquad (14)$$

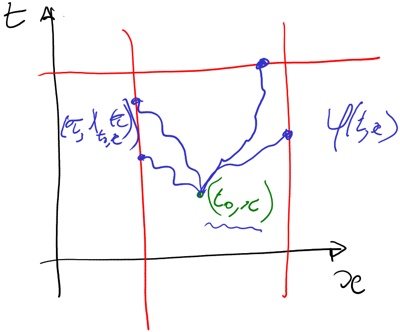
$(t,x) \in Q$, $\tau = \tau_{t,x}$ is the first exit time of $(s, X_{t,x}(s))$ to Γ, $w(s) = (w^1(s), \ldots, w^d(s))^\intercal$ is a standard Wiener process, the $d \times d$ matrix $\sigma(s,x)$ is obtained from $\sigma(s,x)\sigma^\intercal(s,x) = a(s,x)$, $\mu(s,x)$ and $F(s,x)$ are arbitrary d-dimensional vectors sufficiently smooth in $\bar{Q}$.

Cauchy vs Dirichlet problem

Cauchy



Dirichlet



Dirichlet problem: approximation

Weak approximation of stopped diffusions: Milstein (1995), Costantini, Pacchiarotti, Satoretto (1998), Gobet (2000), Milstein, T (2002) and also Springer 2004 or 2021, Gobet, Menozzi (2010)

Apply the weak Euler approximation with the simplest simulation of noise to the system (12)-(14)

$$X_{t,x}(t+h) \approx X = x + h\,(b(t,x) - \sigma(t,x)\,\mu(t,x)) + h^{1/2}\sigma(t,x)\,\xi, \tag{15}$$

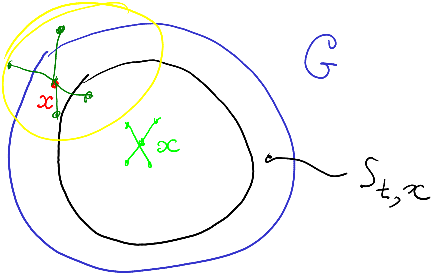
$$Y_{t,x,y}(t+h) \approx Y = y + hc(t,x)\,y + h^{1/2}\mu^{\intercal}(t,x)\,y\,\xi\,, \tag{16}$$

$$Z_{t,x,y,z}(t+h) \approx Z = z + hg(t,x)\,y + h^{1/2}F^{\intercal}(t,x)\,y\,\xi\,, \tag{17}$$

where $\xi = (\xi^1, \dots, \xi^d)^{\intercal}$, ξ^i, $i = 1, \dots, d$, are mutually independent random variables taking the values ± 1 with probability $1/2$.

Dirichlet problem: the simplest random walk

Introduce the set of points close to the boundary (a boundary zone) $S_{t,h} \subset \bar{G}$ on the layer t : we say that $x \in S_{t,h}$ if at least one of the 2^d values of the vector X is outside $\bar{G}$. It is not difficult to see that due to compactness of $\bar{Q}$ there is a constant $\lambda > 0$ such that if the distance from $x \in G$ to the boundary ∂G is equal to or greater than $\lambda\sqrt{h}$ then x is outside the boundary zone and, therefore, for such x all the realizations of the random variable X belong to $\bar{G}$.



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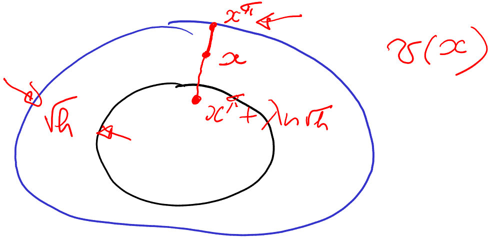
Since restrictions connected with nonexit from the domain $\bar{G}$ should be imposed on an approximation of the system (12), the formulas (15)-(17) can be used only for the points $x \in \bar{G} \backslash S_{t,h}$ on the layer t, and a special construction is required for points from the boundary zone.



Dirichlet problem: the simplest random walk

Let $x \in S_{t,h}$. Denote by $x^\pi \in \partial G$ the projection of the point x on the boundary of the domain G (the projection is unique because h is sufficiently small and ∂G is smooth) and by $n(x^\pi)$ the unit vector of internal normal to ∂G at x^π. Introduce the random vector $X^\pi_{x,h}$ taking two values x^π and $x + h^{1/2}\lambda n(x^\pi)$ with probabilities $p = p_{x,h}$ and $q = q_{x,h} = 1 - p_{x,h}$, respectively, where

$$p_{x,h} = \frac{h^{1/2}\lambda}{|x + h^{1/2}\lambda n(x^\pi) - x^\pi|}\,.$$

If $v(x)$ is a twice continuously differentiable function with the domain of definition $\bar{G}$, then an approximation of $v(x)$ by the expectation $Ev(X^\pi_{x,h})$ corresponds to linear interpolation and

$$v(x) = Ev(X^\pi_{x,h}) + \mathcal{O}(h) = pv(x^\pi) + qv(x + h^{1/2}\lambda n(x^\pi)) + \mathcal{O}(h)\,. \quad (18)$$

We emphasize that the second value $x + h^{1/2}\lambda n(x^\pi)$ does not belong to the boundary zone. We also note that p is always greater than $1/2$ (since the distance from x to ∂G is less than $h^{1/2}\lambda$) and that if $x \in \partial G$ then $p = 1$ (since in this case $x^\pi = x$).

Dirichlet problem: the simplest random walk algorithm

STEP 0. $X'_0 = x_0,\ Y_0 = 1,\ Z_0 = 0,\ k = 0.$

STEP 1. If $X'_k \notin S_{t_k,h}$ then $X_k = X'_k$ and go to STEP 3.
If $X'_k \in S_{t_k,h}$ then either $X_k = X'^{\pi}_k$ with probability $p_{X'_k,h}$ or $X_k = X'_k + h^{1/2}\lambda n(X'^{\pi}_k)$ with probability $q_{X'_k,h}$.

STEP 2. If $X_k = X'^{\pi}_k$ then STOP and $\varkappa = k$,
$X_\varkappa = X'^{\pi}_k,\ Y_\varkappa = Y_k,\ Z_\varkappa = Z_k$.

STEP 3. Simulate ξ_k and find $X'_{k+1},\ Y_{k+1},\ Z_{k+1}$ according to (15)-(17) for $t = t_k,\ x = X_k,\ y = Y_k,\ z = Z_k,$
$\xi = \xi_k$.

STEP 4. If $k + 1 = N$, STOP and $\varkappa = N$, $X_\varkappa = X'_N$, $Y_\varkappa = Y_N$,
$Z_\varkappa = Z_N$, otherwise $k := k + 1$ and return to STEP 1.

Dirichlet problem: the simplest random walk

Theorem

Algorithm has weak order of accuracy $O(h)$, i.e., the inequality

$$|E(\varphi(t_\varkappa, X_\varkappa)\, Y_\varkappa + Z_\varkappa) - u(t_0, x_0)| \leq Ch \tag{19}$$

holds with $C > 0$ independent of t_0, x_0, h.

The scheme of the proof:

- Lemma on order $\mathcal{O}(h^2)$ for the one-step approximation for the Euler approximation.
  The number of steps when $X_k' \notin S_{t_k,h}$ is obviously $\mathcal{O}(1/h)$.
- Lemma on local order $\mathcal{O}(h)$ when X_k' goes outside $\bar{G}$.
- Lemma on the average number of steps when $X_k' \in S_{t_k,h}$ is finite.

Milstein, T (2002) and also Springer 2004 or 2021

Dirichlet problem for elliptic PDE

Consider the Dirichlet problem for elliptic equation

$$\frac{1}{2}\sum_{i,j=1}^{d} a^{ij}(x)\,\frac{\partial^2 u}{\partial x^i \partial x^j} + \sum_{i=1}^{d} b^i(x)\,\frac{\partial u}{\partial x^i} + c(x)\,u + g(x) = 0,\ x \in G\,, \tag{20}$$

$$u\mid_{\partial G} = \varphi(x). \tag{21}$$

The probabilistic representation:

$$u(x) = E\left[\varphi(X_x(\tau))\,Y_{x,1}(\tau) + Z_{x,1,0}(\tau)\right]\,, \tag{22}$$

where $X_x(s)$, $Y_{x,y}(s)$, $Z_{x,y,z}(s)$, $s \geq 0$, is the solution of the Cauchy problem for the system of SDEs:

$$dX = (b(X) - \sigma(X)\mu(X))\,ds + \sigma(X)\,dw(s)\,,\quad X(0) = x, \tag{23}$$

$$dY = c(X)Y\,ds + \mu^{\mathsf{T}}(X)Y\,dw(s)\,,\quad Y(0) = y, \tag{24}$$

$$dZ = g(X)Y\,ds + F^{\mathsf{T}}(X)Y\,dw(s)\,,\quad Z(0) = z, \tag{25}$$

$x \in G$, and $\tau = \tau_x$ is the first exit time of the trajectory $X_x(s)$ to the boundary ∂G.

Dirichlet problem for elliptic PDE

To approximate the solution of the system (23), we construct a Markov chain X_k which stops when it reaches the boundary ∂G at a random step $\varkappa$.

- The simplest random walk is similar to the parabolic case, except $\varkappa$ can be large
- First-order convergence proved.

Milstein, T (2002) and also Springer 2004 or 2021

Robin problem

Let $G \in \mathbb{R}^d$ be a bounded domain with boundary ∂G and $Q := [T_0, T) \times G$ be a cylinder in $\mathbb{R}^{d+1}$.

Consider the Robin problem:

$$\frac{\partial u}{\partial t}+\frac{1}{2}\sum_{i,j=1}^{d} a^{ij}(t,x)\frac{\partial u}{\partial x^i \partial x^j}+\sum_{i=1}^{d} b^i(t,x)\frac{\partial u}{\partial x^i}+c(t,x)u+g(t,x)=0,\ (t,x)\in Q, \tag{26}$$

$$u(T,x)=\varphi(x),\quad x\in\bar{G}, \tag{27}$$

$$\frac{\partial u}{\partial \nu}+\gamma(t,z)u=\psi(t,z),\quad (t,z)\in S, \tag{28}$$

where $\nu = \nu(z)$ is the direction of the inner normal to the surface ∂G at $z \in \partial G$.

Robin problem

The probabilistic representation [Gikhman, Skorohod 1968, Ikeda, Watanabe 1981, Freidlin 1985]:

$$u(t_0, x) = \mathbb{E}\big(\varphi(X_{\mathbf{t_0},x}(T))Y_{\mathbf{t_0},x,1}(T) + Z_{\mathbf{t_0},x,1,0}(T)\big), \tag{29}$$

where $X_{\mathbf{t_0},x}(s)$, $Y_{\mathbf{t_0},x,y}(s)$, $Z_{\mathbf{t_0},x,y,z}(s)$, $s \geq t_0$, is the solution of the system of RSDEs

$$dX(s) = b(s, X(s))ds + \sigma(s, X(s))dW(s) + \nu(X(s))I_{\partial G}(X(s))dL(s), \tag{30}$$

$$dY(s) = c(s, X(s))Y(s)ds + \gamma(s, X(s))I_{\partial G}(X(s))Y(s)dL(s), \tag{31}$$

$$dZ(s) = g(s, X(s))Y(s)ds - \psi(s, X(s))I_{\partial G}(X(s))Y(s)dL(s), \tag{32}$$

with $X(t_0) = x,\ Y(t_0) = y,\ Z(t_0) = z,\ T_0 \leq t_0 \leq s \leq T,\ \ x \in \bar{G}.$

Robin problem

$L(s)$ is the local time of the process $X(s)$ on the boundary ∂G adapted to the filtration $(\mathcal{F}_s)_{s \geq 0}$. A local time is a scalar increasing process continuous in s which increases only when $X(s) \in \partial G$:

$$L(t) = \int_{t_0}^{t} I_{\partial G}\big(X(s)\big)\, dL(s),$$

[Ikeda, Watanabe 1981; P.L. Lions, A.S. Sznitman 1984; Freidlin 1985]

$$L(t) = \int_0^t \delta\big(W(s)\big)\, ds$$

Weak approximation of RSDEs:
Y. Liu (1993); G.N. Milstein (1997); C. Costantini, B. Pacchiarotti, F. Sartoretto (1998); E. Gobet (2001); M. Bossy, E. Gobet, and D. Talay (2004), Leimkuhler, Sharma,T (2022?)

Let $(t_0, x) \in Q$. We introduce the uniform discretization of the time interval $[t_0, T]$ so that $t_0 < \cdots < t_N = T$, $h := (T - t_0)/N$ and $t_{k+1} = t_k + h$.

We consider a Markov chain $(X_k)_{k \geq 0}$ with $X_0 = x$ approximating the solution $X_{t_0,x}(t)$ of the RSDEs

$$\begin{aligned} dX(s) &= b(s, X(s))ds + \sigma(s, X(s))dW(s) + \nu(X(s))I_{\partial G}(X(s))dL(s), \\ X(t_0) &= x. \end{aligned}$$

Since $X(t)$ cannot take values outside $\bar{G}$, the Markov chain should remain in $\bar{G}$ as well. To this end, the chain has an auxiliary (intermediate) step every time it moves from the time layer t_k to t_{k+1}.

Easy-to-implement algorithm

We denote this auxiliary step by $X_{k+1}^{'}$. In moving from X_k to $X_{k+1}^{'}$, we apply the weak Euler scheme

$$X_{k+1}^{'} = X_k + hb_k + h^{1/2}\sigma_k \xi_{k+1}, \qquad (33)$$

where $b_k = b(t_k, X_k)$, $\sigma_k = \sigma(t_k, X_k)$ and $\xi_{k+1} = (\xi_{k+1}^1, \ldots, \xi_{k+1}^d)^\top$, ξ_{k+1}^i, $i = 1, \ldots, d$, $k = 0, \ldots, N-1$, are mutually independent random variables taking values ± 1 with probability $1/2$.

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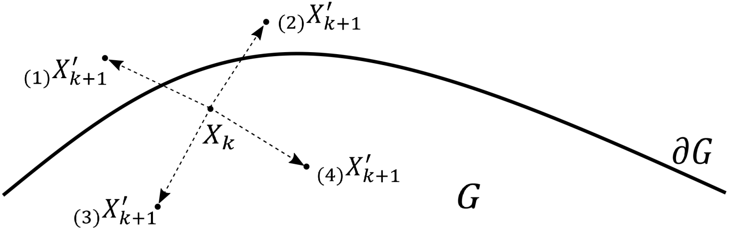
Taking this auxiliary step $X_{k+1}^{'}$ while moving from X_k to X_{k+1} portrays cautious behaviour and gives us an opportunity to check whether the realized value of $X_{k+1}^{'}$ is inside the domain G or not. If $X_{k+1}^{'} \in \bar{G}$ then on the same time layer we assign values to X_{k+1} as

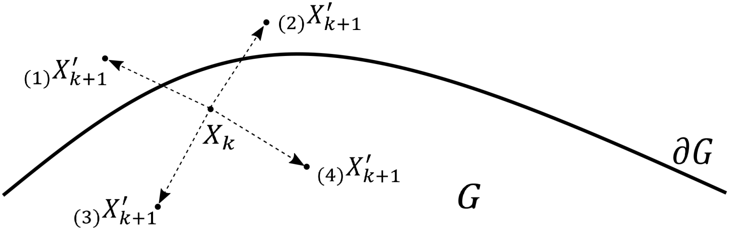
$$X_{k+1} = X_{k+1}^{'}$$

Easy-to-implement algorithm

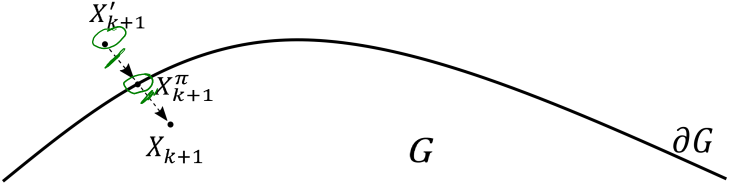


Four possible realizations $_{(i)}X'_{k+1}$ of X'_{k+1} given X_k in two dimensions.

Easy-to-implement algorithm

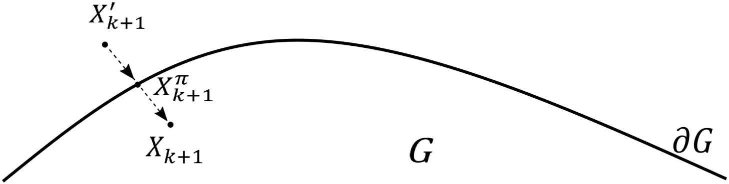


Four possible realizations ${}_{(i)}X^{'}_{k+1}$ of $X^{'}_{k+1}$ given X_k in two dimensions.



One step transition in two dimensions from $X^{'}_{k+1}$ to X_{k+1} using projection X^{π}_{k+1} of $X^{'}_{k+1}$ on ∂G.

Easy-to-implement algorithm



One step transition in two dimensions from $X^{'}_{k+1}$ to X_{k+1} using projection X^{π}_{k+1} of $X^{'}_{k+1}$ on ∂G.

We find the projection of $X^{'}_{k+1}$ onto ∂G which we denote as X^{π}_{k+1} and we calculate $r_{k+1} = \text{dist}(X^{'}_{k+1}, X^{\pi}_{k+1})$ which is the shortest distance between $X^{'}_{k+1}$ and X^{π}_{k+1}. Note that $\text{dist}(X_k, X^{'}_{k+1}) = \mathcal{O}(h^{1/2})$.

$$X_{k+1} = X^{'}_{k+1} + 2r_{k+1}\nu(X^{\pi}_{k+1}). \tag{34}$$

Easy-to-implement algorithm

Algorithm 1 Algorithm to approximate normal reflected diffusion

Step 1: *Set* $X_0 = x$, $X_0^{'} = x$, $k = 0$.

Step 2: *Simulate* ξ_{k+1} *and find* $X_{k+1}^{'}$ *using (33).*

Step 3: ***If*** $X_{k+1}^{'} \in \bar{G}$ *then* $X_{k+1} = X_{k+1}^{'}$, ***else***

- *(i) find the projection* X_{k+1}^{π} *of* $X_{k+1}^{'}$ *on* ∂G,
- *(ii) calculate* $r_{k+1} = \mathrm{dist}(X_{k+1}^{'}, X_{k+1}^{\pi})$ *and find* X_{k+1} *according to (34).*

Step 4: ***If*** $k + 1 = N$ *then* ***stop***, ***else*** *put* $k := k + 1$ *and* ***return*** *to Step 2.*

Easy-to-implement algorithm

We approximate RSDEs (30) according to Algorithm 1 and complement it by an approximation of (31) and (32). If the intermediate step X'_{k+1} introduced in Algorithm 1, belongs to $\bar{G}$ then we use the Euler scheme:

$$Y_{k+1} = Y_k + hc(t_k, X_k)Y_k \tag{35}$$
$$Z_{k+1} = Z_k + hg(t_k, X_k)Y_k. \tag{36}$$

If $X'_{k+1} \notin \bar{G}$ then

$$Y_{k+1} = Y_k + hc(t_k, X_k)Y_k + 2r_{k+1}\gamma(t_{k+1}, X^{\pi}_{k+1})Y_k + 2r^2_{k+1}\gamma^2(t_{k+1}, X^{\pi}_{k+1})Y_k, \tag{37}$$

$$Z_{k+1} = Z_k + hg(t_k, X_k)Y_k - 2r_{k+1}\psi(t_{k+1}, X^{\pi}_{k+1})Y_k \tag{38}$$
$$- 2r^2_{k+1}\psi(t_{k+1}, X^{\pi}_{k+1})\gamma(t_{k+1}, X^{\pi}_{k+1})Y_k,$$

where X^{π}_{k+1} is the projection of X'_{k+1} on ∂G and $r_{k+1} = \mathrm{dist}(X'_{k+1}, X^{\pi}_{k+1})$.

Easy-to-implement algorithm

Algorithm 2 Algorithm to approximate the Robin problem

Step 1: *Set* $X_0 = x$, $Y_0 = 1$, $Z_0 = 0$, $X_0^{'} = x$, $k = 0$.

Step 2: *Simulate* ξ_{k+1} *and find* $X_{k+1}^{'}$ *using (33).*

Step 3: ***If*** $X_{k+1}^{'} \in \bar{G}$ *then* $X_{k+1} = X_{k+1}^{'}$ *and calculate* Y_{k+1} *and* Z_{k+1} *according to (35) and (36), respectively,* ***else*** *find* X_{k+1}, Y_{k+1} *and* Z_{k+1} *according to (34), (37) and (38), respectively.*

Step 4: ***If*** $k + 1 = N$ *then* ***stop***, ***else*** *put* $k := k + 1$ *and* ***return*** *to Step 2.*

Finite-time convergence

Theorem

The weak order of accuracy of the Algorithm is $\mathcal{O}(h)$ under some assumptions, i.e., for sufficiently small $h > 0$

$$|\mathbb{E}(\varphi(X_N)Y_N + Z_N) - u(t_0, X_0)| \ \leq \ Ch, \tag{39}$$

where $u(t,x)$ is solution of (26)-(28) and C is a positive constant independent of h.

The scheme of the proof is roughly as follows.

- Lemma on order $\mathcal{O}(h^2)$ for the one-step approximation for the intermediate step $X_{k+1}^{'}$ (i.e., of the Euler approximation).
  The number of steps when $X_{k+1}^{'} \in \bar{G}$ is obviously $\mathcal{O}(1/h)$.
- Lemma on local order $\mathcal{O}(h^{3/2})$ for X_{k+1} when $X_{k+1}^{'}$ goes outside $\bar{G}$.
- Lemma on the average number of steps when $X_{k+1}^{'} \notin \bar{G}$ is $\mathcal{O}(1/\sqrt{h})$.
  Leimkuhler, Sharma,T (2022?)

Elliptic PDEs with Robin boundary condition

Let $c(x)$ be negative for all $x \in \bar{G}$ and $\gamma(z)$ be non-positive for all $z \in \partial G$. Consider the elliptic equation

$$\frac{1}{2}\sum_{i,j=1}^{d} a^{ij}(x)\frac{\partial^2 u}{\partial x^i \partial x^j} + \sum_{i=1}^{d} b^i(x)\frac{\partial u}{\partial x^i} + c(x)u + g(x) = 0, \quad x \in G, \quad (40)$$

with Robin boundary condition

$$\frac{\partial u}{\partial \nu} + \gamma(z)u = \psi(z), \quad z \in \partial G, \qquad (41)$$

The probabilistic representation [Freidlin 1985]:

$$u(x) = \lim_{T\to\infty} \mathbb{E}\Big(Z_x(T)\Big),$$

where $Z_x(s)$, $x \in \bar{G}$, is governed by the RSDEs

$$dX(s) = b(X(s))ds + \sigma(X(s))dW(s) + \nu(X(s))I_{\partial G}(X(s))dL(s),\ X(0) = x,$$

$$dY(s) = c(X(s))Y(s)ds + \gamma(X(s))I_{\partial G}(X(s))Y(s)dL(s), \quad Y(0) = 1,$$
$$dZ(s) = g(X(s))Y(s)ds - \psi(X(s))I_{\partial G}(X(s))Y(s)dL(s), \quad Z(0) = 0.$$

$\sigma(x)\sigma(x)^\top = a(x).$

Elliptic PDEs with Robin boundary condition

Theorem

Under some assumptions, the following inequality holds for sufficiently small $h > 0$:

$$|\mathbb{E}(Z_N) - u(x)| \leq C\left(h + e^{-\lambda T}\right), \quad (42)$$

where Z_N is calculated according to Algorithm 2 approximating the solution $u(x)$ of (40)-(41), and C and λ are positive constants independent of T and h.

Leimkuhler, Sharma,T (2022?)

Elliptic PDEs with Robin boundary condition

The case $c(x) = 0$ **and** $\gamma(z) = 0$. The probabilistic representation [Freindlin 1985; Bencherif-Madani, Pardoux 2009]:

$$u(x) = \lim_{T \to \infty} \mathbb{E} Z_x(T) + \bar{u}, \tag{43}$$

where $\bar{u} = \int_G u(x)\rho(x)dx$, $\rho(x)$ is the solution of the adjoint problem (note that $\rho(x)$ is the invariant density of $X(s)$), and $Z_x(s) = Z(s)$ is governed by

$$dZ(s) = -\phi_1(X(s))ds - \phi_2(X(s))I_{\partial G}(X(s))dL(s), \;\; Z(0) = 0,$$

where $X(s)$ is as before.

A suitable algorithm based on double partitioning of the time interval $[0, T]$ and its convergence proof are in Leimkuhler, Sharma,T (2022?).

Dirichlet problem for parabolic integro-differential equation

Let G be a bounded domain in $\mathbb{R}^d$, $Q = [t_0, T) \times G$ be a cylinder in $\mathbb{R}^{d+1}$, $\Gamma = \bar{Q} \setminus Q$, $G^c = \mathbb{R}^d \setminus Q$ be the complement of G and $Q^c := (t_0, T] \times G^c \cup \{T\} \times \bar{G}$. Consider the Dirichlet problem for the PIDE:

$$\begin{aligned} \frac{\partial u}{\partial t} + Lu + c(t,x)u + g(t,x) = 0, \quad (t,x) \in Q, \\ u(t,x) = \varphi(t,x), \quad (t,x) \in Q^c, \end{aligned} \tag{44}$$

$$Lu(t,x) := \frac{1}{2} \sum_{i,j=1}^{d} a^{ij}(t,x) \frac{\partial^2 u}{\partial x^i \partial x^j}(t,x) + \sum_{i=1}^{d} b^i(t,x) \frac{\partial u}{\partial x^i}(t,x) \tag{45}$$

$$+ \int_{\mathbb{R}^m} \Big\{ u\big(t, x + F(t,x)z\big) - u(t,x) - \langle F(t,x)z, \nabla u(t,x)\rangle \mathbf{I}_{|z|\le 1} \Big\} \nu(\mathrm{d}z);$$

$F(t,x) = \big(F^{ij}(t,x)\big)$ is a $d \times m$-matrix; and $\nu(z)$, $z \in \mathbb{R}^m$, is a Lévy measure such that $\int_{\mathbb{R}^m} (|z|^2 \wedge 1) \nu(\mathrm{d}z) < \infty$. We allow ν to be of infinite intensity, i.e. $\nu\big(B(0,r)\big) = \infty$ for some $r > 0$, where $B(x,s)$ is the open ball of radius $s > 0$ centred at $x \in \mathbb{R}^d$.

Dirichlet problem for PIDE

Probabilistic representation [Applebaum 2009]

$$u(t,x) = \mathbb{E}\left[\varphi\left(\tau_{t,x}, X_{t,x}(\tau_{t,x})\right) Y_{t,x,1}(\tau_{t,x}) + Z_{t,x,1,0}(\tau_{t,x})\right], \quad (t,x) \in Q, \tag{46}$$

$$dX = b(s, X(s-))\mathrm{d}s + \sigma(s, X(s-))\mathrm{d}w(s) \tag{47}$$

$$+ \int_{\mathbb{R}^d} F(s, X(s-))z\hat{N}(\mathrm{d}z, \mathrm{d}s), \quad X_{t,x}(t) = x,$$

$$dY = c(s, X(s-))Yds, \quad Y_{t,x,y}(t) = y, \tag{48}$$

$$dZ = g(s, X(s-))Yds, \quad Z_{t,x,y,z}(t) = z, \tag{49}$$

and $\tau_{t,x} = \inf\{s \geq t : (s, X_{t,x}(s)) \notin Q\}$ is the fist exit-time of $(s, X_{t,x}(s))$ from Q, $\sigma(s,x)\sigma^\top(s,x) = a(s,x)$; $w(t) = (w^1(t), \dots, w^d(t))^\top$ is a standard d-dimensional Wiener process; and $\hat{N}$ is a Poisson random measure on $[0,\infty) \times \mathbb{R}^m$ with intensity measure $\nu(\mathrm{d}z) \times \mathrm{d}s$, $\int_{\mathbb{R}^m}(|z|^2 \wedge 1)\nu(\mathrm{d}z) < \infty$, and compensated small jumps, i.e.,

$$\hat{N}\left([0,t] \times B\right) = \int_{[0,t]\times B} N(\mathrm{d}z, \mathrm{d}s) - t\nu(B \cap \{|z| \leq 1\}),$$

$$\text{for all } t \geq 0 \text{ and } B \in \mathcal{B}\big(\mathbb{R}^m\big).$$

Dirichlet problem for PIDE

Consider the approximation of (47), where small jumps are replaced by an appropriate diffusion. [Asmussen, Rosinski (2001); Kohatsu-Higa, Tankov (2010); Kohatsu-Higa, Ortiz-Latorre, Tankov (2013); Deligiannidis, Maurer, T (2021)].

Dirichlet problem for PIDE

Consider the approximation of (47), where small jumps are replaced by an appropriate diffusion. [Asmussen, Rosinski (2001); Kohatsu-Higa, Tankov (2010); Kohatsu-Higa, Ortiz-Latorre, Tankov (2013); Deligiannidis, Maurer, T (2021)].
Let γ_ϵ be an m-dimensional vector with the components

$$\gamma_\epsilon^i = \int_{\epsilon \le |z| \le 1} z^i \nu(\mathrm{d}z); \tag{50}$$

and B_ϵ is an $m \times m$ matrix with the components

$$B_\epsilon^{ij} = \int_{|z| < \epsilon} z^i z^j \nu(\mathrm{d}z), \tag{51}$$

while β_ϵ be obtained from the formula $\beta_\epsilon \beta_\epsilon^\top = B_\epsilon$.

Example (Tempered α-stable Process)

For a tempered stable distribution which has Lévy measure given by

$$\nu(\mathrm{d}z) = \Big(\frac{C_+ \mathrm{e}^{-\lambda_+ z}}{z^{1+\alpha}} \mathbf{I}(z > 0) + \frac{C_- \mathrm{e}^{-\lambda_- |z|}}{|z|^{1+\alpha}} \mathbf{I}(z < 0) \Big) \mathrm{d}z,$$

for $\alpha \in (0,2)$ and $C_+,\ C_-,\ \lambda_+,\ \lambda_- > 0$: we find that the error from approximating the small jumps by diffusion as in Theorem is of the order $O(\epsilon^{3-\alpha})$

$$\lambda_\epsilon := \int_{|z|>\epsilon} \nu(\mathrm{d}z) = \mathcal{O}(\epsilon^{-\alpha}), \quad \gamma_\epsilon = \mathcal{O}(\epsilon^{1-\alpha}) \text{ for } \alpha \neq 1 \text{ and } B_\epsilon = \mathcal{O}(\epsilon^{2-\alpha}).$$

Dirichlet problem for PIDE

Consider the modified jump-diffusion $\tilde{X}_{t_0,x}(t) = \tilde{X}^{\epsilon}_{t_0,x}(t)$ defined as

$$\tilde{X}_{t_0,x}(t) = x + \int_{t_0}^{t} \left[b(s, \tilde{X}(s-)) - F(s, \tilde{X}(s-))\gamma_\epsilon \right] \mathrm{d}s + \int_{t_0}^{t} \sigma(s, \tilde{X}(s-)) \mathrm{d}w(s) \tag{52}$$

$$+ \int_{t_0}^{t} F(s, \tilde{X}(s-))\beta_\epsilon \mathrm{d}W(s) + \int_{t_0}^{t} \int_{|z|\geq\epsilon} F(s, \tilde{X}(s-)) z N(\mathrm{d}z, \mathrm{d}s),$$

where $W(t)$ is a standard m-dimensional Wiener process, independent of N and w.

Dirichlet problem for PIDE

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$$+ \int_{t_0}^{t} F(s, \tilde{X}(s-))\beta_\epsilon \mathrm{d}W(s) + \int_{t_0}^{t} \int_{|z|\geq\epsilon} F(s, \tilde{X}(s-))zN(\mathrm{d}z, \mathrm{d}s), \tag{52}$$

where $W(t)$ is a standard m-dimensional Wiener process, independent of N and w.

We observe that, in comparison with (47), in (52) jumps less than ϵ in magnitude are replaced by the additional diffusion part. In this way, the new Lévy measure has finite activity allowing us to simulate its events exactly, i.e. in a practical way.

Consequently,

$$u(t,x) \approx u_\epsilon(t,x) := \mathbb{E}\left[\varphi\left(\tilde{\tau}_{t,x}, \tilde{X}_{t,x}(\tilde{\tau}_{t,x})\right) \tilde{Y}_{t,x,1}(\tilde{\tau}_{t,x}) + \tilde{Z}_{t,x,1,0}(\tilde{\tau}_{t,x})\right], \tag{53}$$

$$(t,x) \in Q,$$

where $\tilde{\tau}_{t,x} = \inf\{s \geq t : (s, \tilde{X}_{t,x}(s)) \notin Q\}$ is the fist exit time of the space-time Lévy process $(s, \tilde{X}_{t,x}(s))$ from the space-time cylinder Q and $\left(\tilde{X}_{t,x}(s), \tilde{Y}_{t,x,y}(s), \tilde{Z}_{t,x,y,z}(s)\right)_{s\geq 0}$ solves the system of SDEs consisting of (52) along with

$$d\tilde{Y} = c(s, \tilde{X}(s-))\tilde{Y}ds, \quad \tilde{Y}_{t,x,y}(t) = y, \tag{54}$$

$$d\tilde{Z} = g(s, \tilde{X}(s-))\tilde{Y}ds, \quad \tilde{Z}_{t,x,y,z}(t) = z. \tag{55}$$

Dirichlet problem for PIDE

Theorem

Under some assumptions, for $0 \le \epsilon < 1$

$$|u^{\epsilon}(t,x) - u(t,x)| \le K \int_{|z| \le \epsilon} |z|^3 \nu(\mathrm{d}z), \quad (t,x) \in Q, \tag{56}$$

where $K > 0$ *does not depend on* t, x, ϵ.

[Deligiannidis, Maurer, T, 2021]

Dirichlet problem for PIDE

Example (Tempered α-stable Process)

For $\alpha \in (0,2)$ and $m = 1$ consider an α-stable process with Lévy measure given by $\nu(\mathrm{d}z) = z^{-1-\alpha}\mathrm{d}z$. Then

$$\int_{|z|\le\epsilon} |z|^3 \nu(\mathrm{d}y) = \frac{\epsilon^{3-\alpha}}{3-\alpha}.$$

Similarly, for a tempered stable distribution which has Lévy measure given by

$$\nu(\mathrm{d}z) = \Big(\frac{C_+ \mathrm{e}^{-\lambda_+ z}}{z^{1+\alpha}} \mathbf{I}(z>0) + \frac{C_- \mathrm{e}^{-\lambda_- |z|}}{|z|^{1+\alpha}} \mathbf{I}(z<0)\Big)\mathrm{d}z,$$

for $\alpha \in (0,2)$ and $C_+,\ C_-,\ \lambda_+,\ \lambda_- > 0$ we find that the error from approximating the small jumps by diffusion as in Theorem is of the order $\mathcal{O}(\epsilon^{3-\alpha})$.

Dirichlet problem for PIDE: Algorithm

Assume that we can exactly sample increments δ between jump times with the intensity

$$\lambda_\epsilon := \int_{|z|>\epsilon} \nu(\mathrm{d}z) \tag{57}$$

and jump sizes J_ϵ are distributed according to the density

$$\rho_\epsilon(z) := \frac{\nu(z)\mathbf{I}_{|z|>\epsilon}}{\lambda_\epsilon}. \tag{58}$$

Dirichlet problem for PIDE: Algorithm

Fix a time-discretization step $h > 0$ and suppose the current position of the chain is (t, x, y, z). If the jump time increment $\delta < h$, we set $\theta = \delta$, otherwise $\theta = h$, i.e. $\theta = \delta \wedge h$.

In the case $\theta = h$, we apply the weak explicit Euler approximation with no jumps:

$$\begin{aligned} \tilde{X}_{t,x}(t+\theta) &\approx X = x + \theta \cdot (b(t,x) - F(t,x)\gamma_\epsilon) \\ &\quad + \sqrt{\theta} \cdot (\sigma(t,x)\,\xi + F(t,x)\beta_\epsilon\, \eta)\,, \end{aligned} \tag{59}$$

$$\tilde{Y}_{t,x,y}(t+\theta) \approx Y = y + \theta \cdot c(t,x)\, y\,, \tag{60}$$

$$\tilde{Z}_{t,x,y,z}(t+\theta) \approx Z = z + \theta \cdot g(t,x)\, y\,, \tag{61}$$

where $\xi = (\xi^1, \ldots, \xi^d)^\intercal$, $\eta = (\eta^1, \ldots, \eta^m)^\intercal$, with $\xi^1, \ldots, \xi^d$ and $\eta^1, \ldots, \eta^m$ mutually independent random variables, taking the values ± 1 with equal probability.

In the case of $\theta < h$, we replace (59) by the following explicit Euler approximation

$$\begin{aligned} \tilde{X}_{t,x}(t+\theta) &\approx X = x + \theta \cdot (b(t,x) - F(t,x)\gamma_\epsilon) \\ &\quad + \sqrt{\theta} \cdot (\sigma(t,x)\,\xi + F(t,x)\beta_\epsilon\, \eta) + F(t,x) J_\epsilon. \end{aligned} \tag{62}$$

Dirichlet problem for PIDE: Algorithm

Let $(t_0, x_0) \in Q$. We aim to find the value $u^\epsilon(t_0, x_0)$. Introduce a discretization of the interval $[t_0, T]$, for example the equidistant one: $h := (T - t_0)/L$.

To approximate the solution of the system (52), we construct a Markov chain $(\vartheta_k, X_k, Y_k, Z_k)$ which stops at a random step $\varkappa$ when (ϑ_k, X_k) exits the domain Q.

Dirichlet problem for PIDE: Algorithm

Initialize: $\vartheta_0 = t_0,\ X_0 = x_0,\ Y_0 = 1,\ Z_0 = 0,\ k = 0.$
while $\vartheta_k < T$ or $X_k \in G$ **do**
    Simulate: ξ_k and η_k with i.i.d. components taking values ± 1 with probability $1/2$ and independently $I_k \sim \mathrm{Bernoulli}\big(1 - e^{-\lambda_\epsilon h}\big)$.
    if $I_k = 0$, **then**
        Set: $\theta_k = h$
        Evaluate: $X_{k+1},\ Y_{k+1},\ Z_{k+1}$ according to (15) – (17).
    else
        Sample: θ_k according to the density $\dfrac{\lambda_\epsilon e^{-\lambda_\epsilon x}}{1 - e^{-\lambda_\epsilon h}}$.
        Sample: jump size $J_{\epsilon,k}$ according to the density (58).
        Evaluate: $X_{k+1},\ Y_{k+1}$, Z_{k+1} according to (18), (16), (17).
    end if
    Set: $\vartheta_{k+1} = \vartheta_k + \theta_k$ and $k = k + 1$.
end while
Set: $X_\varkappa = X_k,\ Y_\varkappa = Y_k,\ Z_\varkappa = Z_k,\ \varkappa = k,\ \vartheta_\varkappa = \vartheta_k.$
if $\vartheta_\varkappa < T$ **then Set:** $\bar{\vartheta}_\varkappa = \vartheta_\varkappa$
else Set: $\bar{\vartheta}_\varkappa = T$
end if

Dirichlet problem for PIDE: Algorithm

Theorem

Under some assumption, the global error of the Algorithm satisfies the following bound

$$\begin{aligned} &\left|\mathbb{E}[\varphi(\bar{\vartheta}_\varkappa, X_\varkappa)Y_\varkappa + Z_\varkappa] - u^\epsilon(t_0, x_0)\right| \\ \leq\ & K(1+\gamma_\epsilon^2)\left(\frac{1}{\lambda_\epsilon} - h\frac{e^{-\lambda_\epsilon h}}{1-e^{-\lambda_\epsilon h}}\right) + K\frac{1-e^{-\lambda_\epsilon h}}{\lambda_\epsilon}, \end{aligned} \tag{63}$$

where $K > 0$ is a constant independent of h and ϵ.

[Deligiannidis, Maurer, T, 2021]

$$\lambda_\epsilon = \int_{|z|>\epsilon} \nu(\mathrm{d}z)$$

Dirichlet problem for PIDE: Algorithm

If $\lambda_\epsilon h < 1$, we obtain:

$$\left|\mathbb{E}[\varphi(\bar{\vartheta}_\varkappa, X_\varkappa) Y_\varkappa + Z_\varkappa] - u^\epsilon(t_0, x_0)\right| \leq K(1 + |\gamma_\epsilon|^2) h,$$

which is expected for weak convergence in the jump-diffusion case.
If λ_ϵ is large (meaning that almost always $\theta < h$), the error is tending to

$$\left|\mathbb{E}[\varphi(\bar{\vartheta}_\varkappa, X_\varkappa) Y_\varkappa + Z_\varkappa] - u^\epsilon(t_0, x_0)\right| \leq K(1 + |\gamma_\epsilon|^2) \frac{1}{\lambda_\epsilon}.$$

We also remark that for any fixed λ_ϵ, we have first order convergence when $h \to 0$.

Dirichlet problem for PIDE: Algorithm

$$\left|\mathbb{E}[\varphi(\bar{\vartheta}_{\varkappa}, X_{\varkappa})Y_{\varkappa} + Z_{\varkappa}] - u(t_0, x_0)\right| \tag{64}$$
$$\leq K(1+|\gamma_\epsilon|^2)\left(\frac{1}{\lambda_\epsilon} - h\frac{e^{-\lambda_\epsilon h}}{1-e^{-\lambda_\epsilon h}}\right) + K\frac{1-e^{-\lambda_\epsilon h}}{\lambda_\epsilon} + K\int_{|z|\leq\epsilon}|z|^3\nu(\mathrm{d}z).$$

- For $\alpha \in (0,1)$ convergence is linear in cost and there is no benefit of restricting jump adapted steps by h.
- For $\alpha \in (1,2)$, it is beneficial to use restricted jump-adapted steps to get the order of $(3-\alpha)/(1+\alpha)$ in cost.
- Restricted jump-adapted steps should typically be used for jump-diffusions (the finite activity case when there is no singularity of λ_ϵ and γ_ϵ) because jump time increments δ typically take too large values and to control the error at every step we should truncate those times at a sufficiently small $h > 0$ for a satisfactory accuracy.

Conclusions

• Using probabilistic representations, we can approximate various problems for parabolic and elliptic 2nd order PDEs and PIDEs.

• We considered simplest (and hence easy to implement) algorithms for:

- Dirichlet problems for linear parabolic and eliptic PDEs
- Robin problems for linear parabolic and eliptic PDEs
- Dirichlet problem for linear parabolic PIDE